Asymptotic windings of the unitary Brownian motion

Fabrice Baudoin (Joint with J. Wang)

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Introduction: Winding form on the plane

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$$\gamma(t) = |\gamma(t)| \exp\left(i \int_{\gamma[0,t]} \alpha \right), \quad t \geq 0.$$

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It is therefore natural to call α the winding form around 0 since the integral of a path γ along this form quantifies the angular motion of this path.

Brownian Winding

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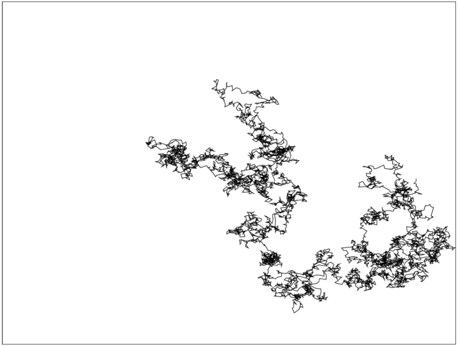

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The integral of the winding form along the paths of a two-dimensional Brownian motion $Z(t) = X(t) + iY(t)$ which is not started from 0 can be studied using Itô's calculus (or rough paths theory) and yields the Brownian winding functional:

$$\zeta(t) = \int_{Z[0,t]} \alpha = \int_0^t \frac{X(s)dY(s) - Y(s)dX(s)}{X(s)^2 + Y(s)^2}.$$

Spitzer theorem

Theorem (Spitzer, 1958)

When $t \to +\infty$, in distribution

$$\frac{2}{\ln t}\zeta(t) \to C_1$$

where C_1 is a Cauchy distribution with parameter 1.

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The characteristic function of ζ can therefore be expressed in terms of the Laplace transform of the additive functional $\int_0^t \frac{ds}{R_s^2}$.

Spitzer theorem

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In this talk, we shall be interested in windings associated with unitary Brownian motions.

Windings of unitary Brownian motions

Theorem (Baudoin-Wang, EJP 2021)

Let

$$U_t = \begin{pmatrix} X_t & Y_t \\ Z_t & W_t \end{pmatrix}$$

be a Brownian motion on the unitary group $\mathbf{U}(n)$ *with* $Z_t \in \mathbb{C}^{k\times k}$*,* $1 \leq k \leq n-1$*.*

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be a Brownian motion on the unitary group $\mathbf{U}(n)$ *with* $Z_t \in \mathbb{C}^{k\times k}$, $1 \leq k \leq n-1$. *Assume that* $\det Z_0 \neq 0$. *One has then the polar decomposition*

$$\det(Z_t) = \varrho_t e^{i\theta_t}$$

where $0 < \varrho_t \leq 1$ *and* θ_t *is a real-valued continuous martingale such that the following convergence holds in distribution when* $t \to +\infty$

$$\frac{\theta_t}{t} \to \mathcal{C}_{k(n-k)},$$

where $\mathcal{C}_{k(n-k)}$ *is a Cauchy distribution of parameter* $k(n-k)$.

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The second step is to implement Yor's Girsanov transform method in this context.

Stiefel fibration

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The complex Stiefel manifold $V_{n,k}$ is the set of unitary k-frames in $\mathbb{C}^n$:

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When $k = 1$, $V_{n,k} \simeq \mathbb{S}^{2n-1}$, $G_{n,1} \simeq \mathbb{CP}^{n-1}$ and the Stiefel fibration reduces to the Hopf fibration.

Block decomposition of the unitary Brownian motion

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The process $(U_t)_{t \geq 0}$ is called a Brownian motion on $\mathbf{U}(n)$.

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Theorem

Let $U_t = \begin{pmatrix} X_t & Y_t \\ Z_t & V_t \end{pmatrix}$, $t \geq 0$ *be a Brownian motion on* $\mathbf{U}(n)$ *with* $Z_0 \in GL(k, \mathbb{C})$.

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1. *The process* $\begin{pmatrix} X_t \\ Z_t \end{pmatrix}_{t \geq 0}$ *is a Brownian motion on the Stiefel manifold* $V_{n,k}$ *;*
2. *The process* $(w_t)_{t \geq 0} := (X_t Z_t^{-1})_{t \geq 0}$ *is a Brownian motion on the complex Grasmannian* $G_{n,k}$ *.*

Skew-product decomposition of the Stiefel Brownian motion

Using the Stiefel fibration

$$\mathbf{U}(k) \to V_{n,k} \to G_{n,k}$$

this yields a skew-product decomposition of the process $\begin{pmatrix} X_t \\ Z_t \end{pmatrix}_{t \geq 0}$ in terms of a Brownian motion w on $G_{n,k}$ and a Brownian motion on $\mathbf{U}(k)$.

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Skew-product decomposition of the Stiefel Brownian motion

Theorem

Let $(w_t)_{t\geq 0}$ be a Brownian motion on $G_{n,k}$ and let $(\Omega_t)_{t\geq 0}$ be a Brownian motion on the unitary group $\mathbf{U}(k)$ independent from $(w_t)_{t\geq 0}$. Let $(\Theta_t)_{t\geq 0}$ be the $\mathbf{U}(k)$-valued solution of the Stratonovich stochastic differential equation

$$\begin{cases} d\Theta_t = \circ d\mathfrak{a}_t\, \Theta_t \\ \Theta_0 = (Z_0 Z_0^*)^{-1/2} Z_0, \end{cases}$$

where $\mathfrak{a}_t = \int_{w[0,t]} \eta$. The process

$$\begin{pmatrix} w_t \\ I_k \end{pmatrix} (I_k + w_t^* w_t)^{-1/2} \Theta_t\, \Omega_t$$

is a Brownian motion on $V_{n,k}$ started at $\begin{pmatrix} w_0 Z_0 \\ Z_0 \end{pmatrix}$.

Skew-product decomposition of the Stiefel Brownian motion

The $\mathfrak{u}(k)$-valued one-form η is given by

$$\eta := \frac{1}{2}\Big((I_k + w^*w)^{-1/2}(dw^*\, w - w^* dw)(I_k + w^*w)^{-1/2} + [d(I_k + w^*w)^{1/2}, (I_k + w^*w)^{-1/2}]\Big)\,.$$

Skew-product decomposition of the Stiefel Brownian motion

From the decomposition

$$\det(Z_t) = \det(I_k + w_t^* w_t)^{-1/2} \det \Theta_t \, \det \Omega_t$$

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one deduces that

$$\det(Z_t) = \varrho_t e^{i\theta_t}$$

with

$$\varrho_t = \det(I_k + J_t)^{-1/2}, \ \ i\theta_t = i\theta_0 + \mathrm{tr}(D_t) + \int_{w[0,t]} \mathrm{tr}(\eta)$$

where D_t is a Brownian motion on $\mathfrak{u}(k)$ independent from w and θ_0 is such that $e^{i\theta_0} = \frac{\det Z_0}{|\det Z_0|}$.

Winding functional

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Lemma

We have

$$\int_{w[0,t]} \mathrm{tr}(\eta) = i\mathcal{B}_{\int_0^t \mathrm{tr}(w_s^* w_s) ds}$$

where $\mathcal{B}$ is a one-dimensional Brownian motion independent from the process $\mathrm{tr}(w^* w)$.

Limit theorem

The main limit theorem is then the following.

Theorem

The following convergence holds in distribution when $t \to +\infty$

$$\frac{1}{t^2} \int_0^t \mathrm{tr}(w_s^* w_s)\, ds \to X,$$

where X *is a random variable on* $[0, +\infty)$ *with density*

$$\frac{k(n-k)}{\sqrt{2\pi} x^{3/2}} e^{-\frac{k^2(n-k)^2}{2x}}$$

.

Laplace transform

The proof of the theorem relies on an explicit formula for the Laplace transform of $\int_0^t \mathrm{tr}(w_s^* w_s)\, ds$ which is obtained using Yor's method and the Karlin-McGregor formula.

Lemma

For every $\alpha \geq 0$ *and* $t > 0$

$$\mathbb{E}\left(e^{-2\alpha^2 \int_0^t \mathrm{tr}(w_s^* w_s) ds}\right)$$

$$= C e^{\left(\frac{1}{3}k(k-1)(3n-4k+6\alpha+2)-2k(n-k)\alpha\right)t}$$

$$\int_{\Delta_k} \det\left(\frac{p_t^{n-2k,2\alpha}\left(\frac{1-\lambda_i(0)}{1+\lambda_i(0)}, x_j\right)}{(1+x_j)^\alpha}\right)_{i,j} \prod_{i>j}(x_i - x_j)\, dx,$$

where the λ_i*'s are the eigenvalues of* $w^* w$*.*

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$$M_t^\alpha = e^{2k\alpha(n-k)t} \left(\frac{\det(I_k + J_0)}{\det(I_k + J_t)} \right)^\alpha \exp\left(-2\alpha^2 \int_0^t \mathrm{tr}(J) ds \right)$$

is a martingale.

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is a martingale. Consider now the probability measure P^α defined by

$$P^\alpha|_{\mathcal{F}_t} = M_t^\alpha \cdot P|_{\mathcal{F}_t}.$$

Laplace transform

We first note that

$$\mathbb{E}\left(e^{-2\alpha^2\int_0^t \mathrm{tr}(J)ds}\right) = e^{-2k(n-k)\alpha t}\,\mathbb{E}^{\alpha}\left[\left(\frac{\det(I_k+J_t)}{\det(I_k+J_0)}\right)^{\alpha}\right].$$

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and then that under P^{α}, J is still a matrix Jacobi process but with different parameters. One concludes with Karlin-McGregor formula which yields the density of the eigenvalues of matrix Jacobi processes.

Asymptotics of the radial motions

Interestingly, our analysis also yields that when $t \to +\infty$,

$$|\det Z_t|^2 \to \prod_{j=1}^{\min(k,n-k)} \mathfrak{B}_{j,\max(k,n-k)}$$

where $\mathfrak{B}_{a,b}$ are independent beta random variables with parameters (a, b).

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where $\mathfrak{B}_{a,b}$ are independent beta random variables with parameters (a, b). However, since $Z^* Z$ is a matrix Jacobi process, this last result can be more easily obtained using results on the Jacobi ensemble by A. Rouault.