High-dimensional, multiscale online changepoint detection

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DataSig seminar

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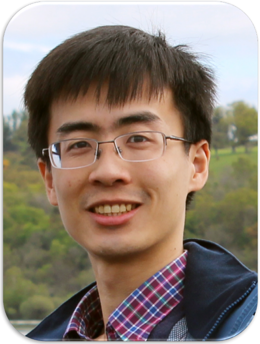
Collaborators

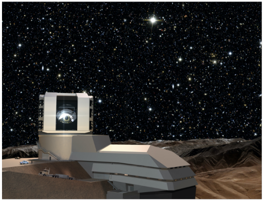
Yudong Chen



Tengyao Wang

Changepoint problems

- Modern technology has facilitated the real-time monitoring of many types of evolving processes.



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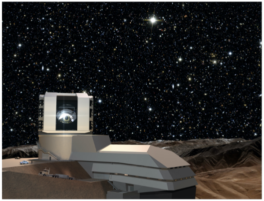

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- Very often, a key feature of interest for data streams is a changepoint.

From offline to online

- The vast majority of the changepoint literature concerns the offline problem (Killick et al., 2012; Wang and Samworth, 2018; Wang et al., 2018; Baranowski et al., 2019; Liu, Gao and Samworth, 2021).

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- Univariate online changepoints have been studied within the well-established field of *statistical process control* (Duncan, 1952; Page, 1954; Barnard, 1959; Fearnhead and Liu, 2007; Oakland, 2007).
- Much less work on multivariate, online changepoint problems (Tartakovsky et al., 2006; Mei, 2010; Zou et al., 2015). Several methods involve scanning a moving window of fixed size for changes (Xie and Siegmund, 2013; Soh and Chandrasekaran, 2017; Chan, 2017).

Online algorithm

Key definition of an online algorithm:

Definition. The computational complexity for processing a new observation, and the storage requirements, depend only on **the number of bits needed to represent the new data**.

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- For the purposes of this definition, all real numbers are considered as floating point numbers.
- Importantly, the computational complexity is not allowed to depend on the number of previously observed data points.

Problem setting

We consider a high-dimensional online changepoint detection problem for independent random vectors $(X_n)_{n\in\mathbb{N}}$:

- **Data generating mechanism:** for some unknown, deterministic time $z \in \mathbb{N} \cup \{0\}$, we have

$$X_1, \ldots, X_z \sim N_p(0, I_p) \quad \text{and} \quad X_{z+1}, X_{z+2}, \ldots \sim N_p(\theta, I_p).$$

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- $\theta \neq 0$: data generated **under the alternative**, i.e. there exists a change.

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- $\theta = 0$: data generated **under the null**, i.e. no change.
- $\theta \neq 0$: data generated **under the alternative**, i.e. there exists a change.

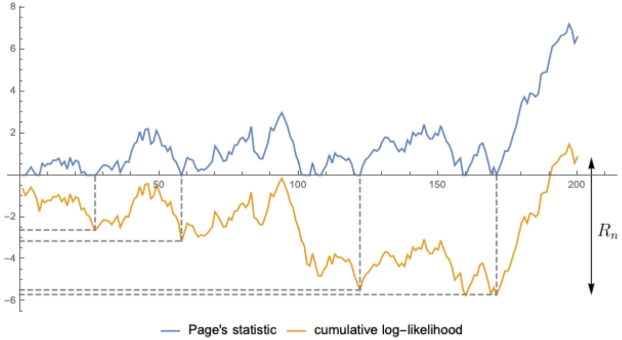
- Assume $\vartheta := \|\theta\|_2$ is at least a known lower bound $\beta > 0$.

Example of an online algorithm (Page, 1954)

Let $p = 1$ and assume $\theta > 0$. Page's procedure:

$$R_n := \max_{0 \le h \le n} \sum_{i=n-h+1}^{n} \beta(X_i - \beta/2) = \max\{R_{n-1} + \beta(X_n - \beta/2), 0\}.$$

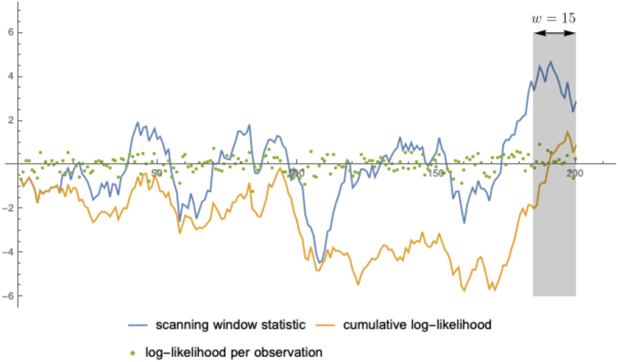
Threshold $T \equiv T_\beta$ for changepoint declaration.



Example of an online algorithm?

Let $p = 1$ and assume $\theta > 0$. Scanning window-based method with window width $w > 0$:

$$W_n := \sum_{i=n-w+1}^{n} \beta(X_i - \beta/2).$$



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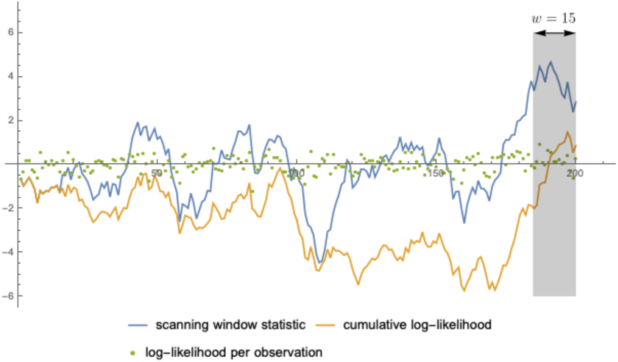

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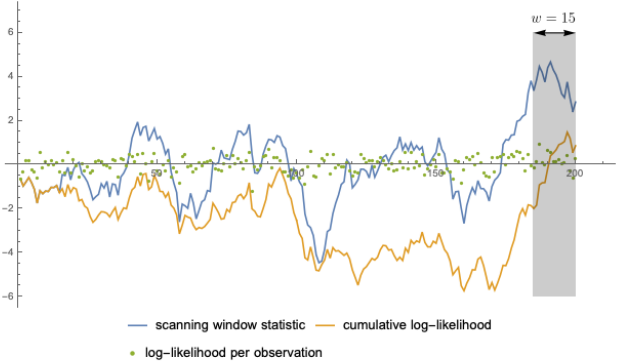
- Window size w needs to increase when β decreases.
- Computational complexity depends on β.



Procedures and performance measures

A *sequential changepoint procedure* is an extended stopping time N (w.r.t. the natural filtration) taking values in $\mathbb{N} \cup \{\infty\}$.

- The *patience* of a sequential changepoint procedure N is $\mathbb{E}_0(N)$; also known as the average run length to false alarm.

- Two types of *response delays*:
  - (Average case) response delay
$$\bar{\mathbb{E}}_\theta(N) := \sup_{z\in\mathbb{N}} \mathbb{E}_{z,\theta}\big\{(N-z)\vee 0\big\};$$
  - Worst case response delay
$$\bar{\mathbb{E}}_\theta^{\mathrm{wc}}(N) := \sup_{z\in\mathbb{N}} \operatorname{ess\,sup} \mathbb{E}_{z,\theta}\big\{(N-z)\vee 0 \mid X_1,\ldots,X_z\big\}.$$

  Thus,
$$\bar{\mathbb{E}}_\theta(N) \le \bar{\mathbb{E}}_\theta^{\mathrm{wc}}(N).$$

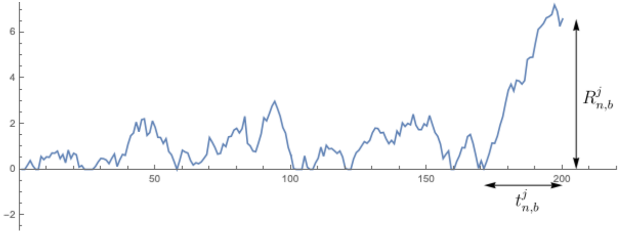
A high-dimensional, multiscale online algorithm: `ocd`

Diagonal statistics

- Write $X_i = (X_i^1, \ldots, X_i^p)^\top \in \mathbb{R}^p$. For $n \in \mathbb{N}$, $b \in \mathbb{R}\setminus\{0\}$ and $j \in [p]$, define

$$R_{n,b}^j := \max_{0\le h\le n} \sum_{i=n-h+1}^{n} b(X_i^j - b/2)$$

$$t_{n,b}^j := \operatorname*{argmax}_{0\le h\le n} \sum_{i=n-h+1}^{n} b(X_i^j - b/2).$$

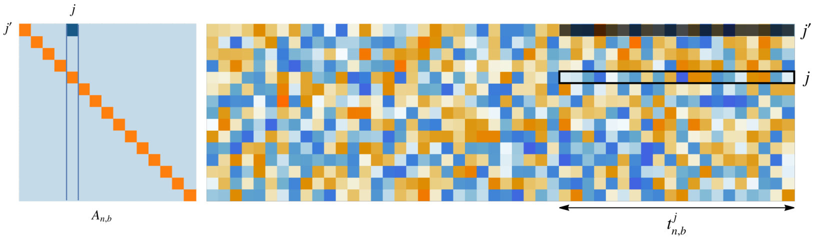


- $(R_{n,b}^j)_{j\in[p]}$ are called the **diagonal statistics**.

Off-diagonal statistics

- For each $j \in [p]$, compute tail partial sums of length $t^j_{n,b}$ in all coordinates $j' \in [p]$:

$$A^{j',j}_{n,b} := \sum_{i=n-t^j_{n,b}+1}^{n} X^{j'}_i .$$



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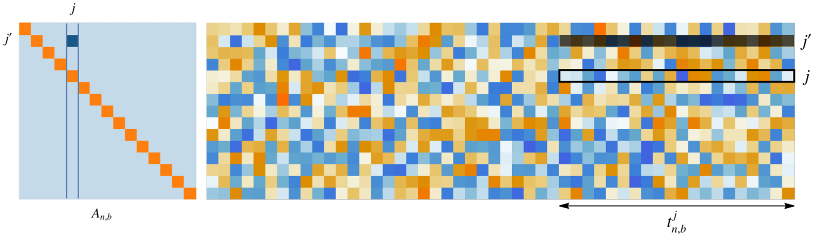

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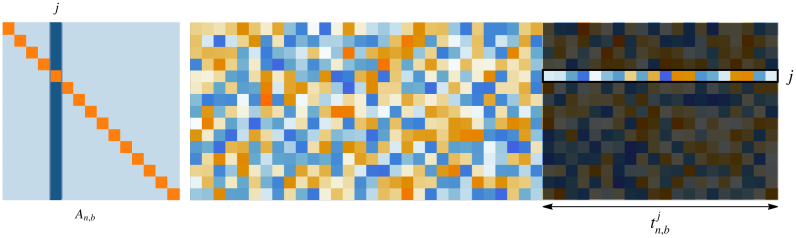

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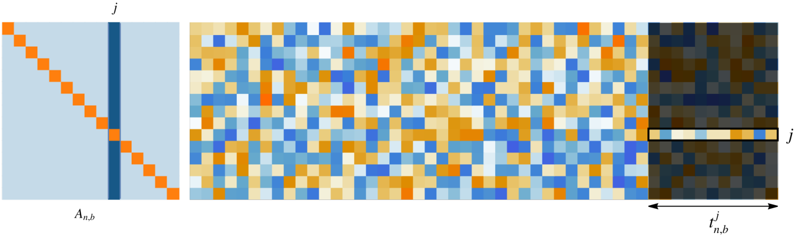

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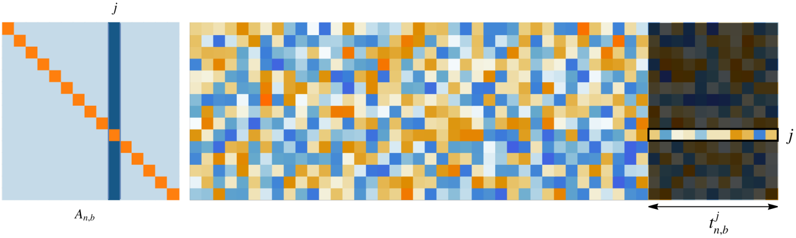

- We aggregate to form an **off-diagonal statistic** anchored at coordinate j:

$$Q^j_{n,b} := \sum_{j' \in [p]: j' \neq j} \frac{(A^{j',j}_{n,b})^2}{t^j_{n,b} \vee 1} \mathbb{1}_{\left\{|A^{j',j}_{n,b}| \geq a\sqrt{t^j_{n,b}}\right\}} .$$

- Different values of a can be chosen to detect dense or sparse signals.

Aggregation

- Allow b to range over a (signed) dyadic grid $\mathcal{B} \cup \mathcal{B}_0$, where

$$\mathcal{B} := \left\{ \pm \frac{\beta}{\sqrt{2^{\ell} \log_2(2p)}} : \ell = 0, \ldots, \lfloor \log_2(p) \rfloor \right\},$$

$$\mathcal{B}_0 := \left\{ \pm \frac{\beta}{\sqrt{2^{\lfloor \log_2(2p) \rfloor} \log_2(2p)}} \right\}.$$

- Aggregate diagonal statistics:

$$\begin{aligned} S_n^{\mathrm{diag}} &:= \max_{(j,b) \in [p] \times (\mathcal{B} \cup \mathcal{B}_0)} R_{n,b}^{j} \\ &= \max_{(j,b) \in [p] \times (\mathcal{B} \cup \mathcal{B}_0)} \bigl(b A_{n,b}^{j,j} - b^2 t_{n,b}^{j}/2 \bigr). \end{aligned}$$

- Aggregate off-diagonal statistics

$$S_n^{\mathrm{off}} := \max_{(j,b) \in [p] \times \mathcal{B}} Q_{n,b}^{j}.$$

- Declare change when either S_n^{diag} or S_n^{off} is large.

Pseudocode

Algorithm 1: Pseudo-code of the `ocd` algorithm

Input: $X_1, X_2 \ldots \in \mathbb{R}^p$ observed sequentially, $\beta > 0$, $a \geq 0$, $T^{\mathrm{diag}} > 0$ and $T^{\mathrm{off}} > 0$

Set: $\mathcal{B} = \left\{ \pm \frac{\beta}{\sqrt{2^\ell \log_2(2p)}} : \ell = 0, \ldots, \lfloor \log_2 p \rfloor \right\}$, $\mathcal{B}_0 = \left\{ \pm \frac{\beta}{\sqrt{2^{\lfloor \log_2 p \rfloor + 1} \log_2(2p)}} \right\}$, $n = 0$,

$A_b = \mathbf{0} \in \mathbb{R}^{p \times p}$ and $t_b = 0 \in \mathbb{R}^p$ for all $b \in \mathcal{B} \cup \mathcal{B}_0$

repeat

- $n \leftarrow n + 1$
- observe new data vector X_n
- **for** $(j, b) \in [p] \times (\mathcal{B} \cup \mathcal{B}_0)$ **do**
  - $t_b^j \leftarrow t_b^j + 1$
  - $A_b^{\cdot, j} \leftarrow A_b^{\cdot, j} + X_n$
  - **if** $b A_b^{j,j} - b^2 t_b^j / 2 \leq 0$ **then**
    - $t_b^j \leftarrow 0$ and $A_b^{\cdot, j} \leftarrow 0$
  - compute $Q_b^j \leftarrow \sum_{j' \in [p] : j' \neq j} \frac{(A_b^{j',j})^2}{t_b^j \vee 1} \mathbb{1}_{\left\{ |A_b^{j',j}| \geq a \sqrt{t_b^j} \right\}}$
- $S^{\mathrm{diag}} \leftarrow \max_{(j,b) \in [p] \times (\mathcal{B} \cup \mathcal{B}_0)} \left(b A_b^{j,j} - b^2 t_b^j / 2 \right)$
- $S^{\mathrm{off}} \leftarrow \max_{(j,b) \in [p] \times \mathcal{B}} Q_b^j$

until $S^{\mathrm{diag}} \geq T^{\mathrm{diag}}$ or $S^{\mathrm{off}} \geq T^{\mathrm{off}}$;

Output: $N = n$

Computational Complexity: $O(p^2 \log(ep))$

Dense, sparse and adaptive versions

- Dense change: choose $a = a^{\text{dense}} = 0$, and let $S^{\text{off,d}} = S^{\text{off}}\left(a^{\text{dense}}\right)$.
- Sparse change: choose $a = a^{\text{sparse}} = \sqrt{8\log(p-1)}$, and let $S^{\text{off,s}} = S^{\text{off}}\left(a^{\text{sparse}}\right)$.

- We combine the two cases to form an adaptive procedure, which has output $N = \min\left\{N^{\text{diag}}, N^{\text{off,d}}, N^{\text{off,s}}\right\}$, where

$$N^{\text{diag}} := \inf\{n : S_n^{\text{diag}} \geq T^{\text{diag}}\}$$
$$N^{\text{off,d}} := \inf\{n : S_n^{\text{off,d}} \geq T^{\text{off,d}}\}$$
$$N^{\text{off,s}} := \inf\{n : S_n^{\text{off,s}} \geq T^{\text{off,s}}\},$$

for some thresholds $T^{\text{diag}}, T^{\text{off,d}}$ and $T^{\text{off,s}}$.

Why does ocd work?

- Patience $\mathbb{E}_0(N)$ can be guaranteed by choosing thresholds $T^{\mathrm{diag}}, T^{\mathrm{off,d}}$ and $T^{\mathrm{off,s}}$ appropriately.

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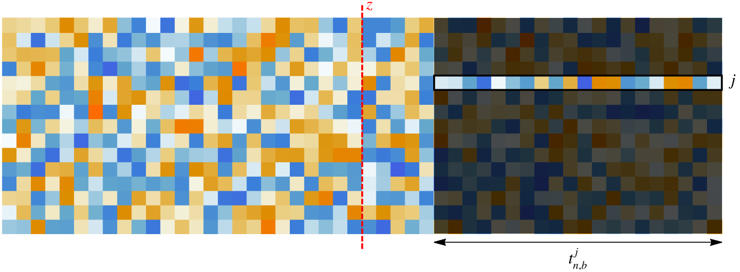
- Patience $\mathbb{E}_0(N)$ can be guaranteed by choosing thresholds $T^{\mathrm{diag}}, T^{\mathrm{off,d}}$ and $T^{\mathrm{off,s}}$ appropriately.

- Diagonal statistics are useful for detecting changes whose signal is concentrated in one or few coordinates.
- Off-diagonal statistics are useful in detecting changes whose signal is not highly concentrated.

A slight variant of `ocd`

- Instead of aggregating over the last $t^j_{n,b}$ points, we would like to aggregate over $\approx t^j_{n,b}/2$ points to form off-diagonal statistics $Q^j_{n,b}$.



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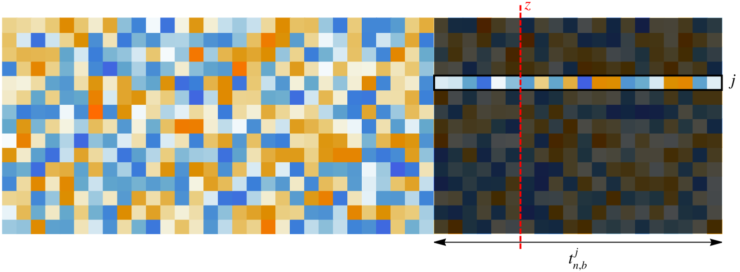

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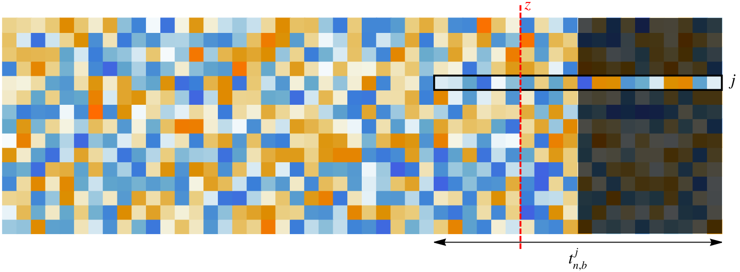

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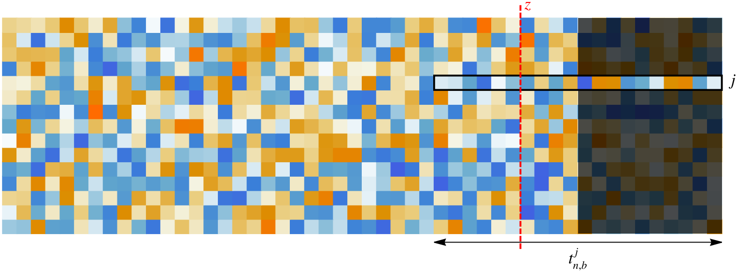

How can we achieve this in an online manner?

Given a sequence of real observations $(X_t)_{t\in\mathbb{N}}$, how can we keep track of the sum of the final $\tau \approx t/2$ observations at time t in an online way?

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t	1	2	3	4	5	6	7	8	...
$\boldsymbol{\tau}$	1	1	2	2	3	4	5	4	...
$\boldsymbol{\Lambda}$	X_1	X_2	X_2+X_3	X_3+X_4	$X_3+X_4+X_5$	$X_3+\cdots+X_6$	$X_3+\cdots+X_7$	$X_5+\cdots+X_8$	...
$\tilde{\boldsymbol{\Lambda}}$	0	0	X_3	0	X_5	X_5+X_6	$X_5+X_6+X_7$	0	...

$t/2 \le \tau < 3t/4$ for $t \ge 2$.

(Part of) modified algorithm: ocd′

for $(j,b) \in [p] \times (\mathcal{B} \cup \mathcal{B}_0)$ **do**

$t_b^j \leftarrow t_b^j + 1$ and $A_b^{\cdot,j} \leftarrow A_b^{\cdot,j} + X_n$

set $\delta = 0$ if t_b^j is a power of 2 and $\delta = 1$ otherwise.

$\tau_b^j \leftarrow \tau_b^j\delta + \tilde{\tau}_b^j(1-\delta) + 1$ and $\Lambda_b^{\cdot,j} \leftarrow \Lambda_b^{\cdot,j}\delta + \tilde{\Lambda}_b^{\cdot,j}(1-\delta) + X_n$

$\tilde{\tau}_b^j \leftarrow (\tilde{\tau}_b^j + 1)\delta$ and $\tilde{\Lambda}_b^{\cdot,j} \leftarrow (\tilde{\Lambda}_b^{\cdot,j} + X_n)\delta$.

if $bA_b^{j,j} - b^2 t_b^j/2 \le 0$ **then**

$t_b^j \leftarrow \tau_b^j \leftarrow \tilde{\tau}_b^j \leftarrow 0$

$A_b^{\cdot,j} \leftarrow \Lambda_b^{\cdot,j} \leftarrow \tilde{\Lambda}_b^{\cdot,j} \leftarrow 0$

compute $Q_b^j \leftarrow \sum_{j'\in[p]:j'\neq j} \frac{(\Lambda_b^{j',j})^2}{\tau_b^j \vee 1} \mathbb{1}_{\{|\Lambda_b^{j',j}| \ge a\sqrt{\tau_b^j}\}}$

Theoretical guarantees: patience

Choose thresholds

$$T^{\mathrm{diag}} = \log\{24p\gamma \log_2(4p)\}$$
$$T^{\mathrm{off,d}} = \psi\big(2\log\{24p\gamma \log_2(2p)\}\big)$$
$$T^{\mathrm{off,s}} = 8\log\{24p\gamma \log_2(2p)\}$$

where $\psi(x) := p - 1 + x + \sqrt{2(p-1)x}$ and $\gamma \geq 1$ is a user-specified desired patience level.

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where $\psi(x) := p - 1 + x + \sqrt{2(p-1)x}$ and $\gamma \geq 1$ is a user-specified desired patience level.

Theorem. Assume there is no change. Then, the adaptive version of ocd′ with the above choice of thresholds satisfies $\mathbb{E}_0(N) \geq \gamma$.

Theoretical guarantees: response delay

Effective sparsity of $\theta \in \mathbb{R}^p$: smallest $s \equiv s(\theta) \in \{2^0, 2^1, \ldots, 2^{\lfloor \log_2 p \rfloor}\}$ such that

$$\left| \left\{ j \in [p] : |\theta^j| \geq \frac{\|\theta\|_2}{\sqrt{s(\theta) \log_2(2p)}} \right\} \right| \geq s(\theta).$$

Theoretical guarantees: response delay

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$$\left| \left\{ j \in [p] : |\theta^j| \geq \frac{\|\theta\|_2}{\sqrt{s(\theta) \log_2(2p)}} \right\} \right| \geq s(\theta).$$

Theorem. Assume that change happens at time z and that the post-change signal θ satisfies $\|\theta\|_2 = \vartheta \geq \beta > 0$ with effective sparsity s. Then, the adaptive version of ocd$'$ with the same choice of thresholds satisfies:

(a) (Worst case response delay)

$$\bar{\mathbb{E}}_\theta^{\mathrm{wc}}(N) \lesssim \frac{s \log(ep\gamma) \log(ep)}{\beta^2} \vee 1;$$

(b) (Average case response delay)

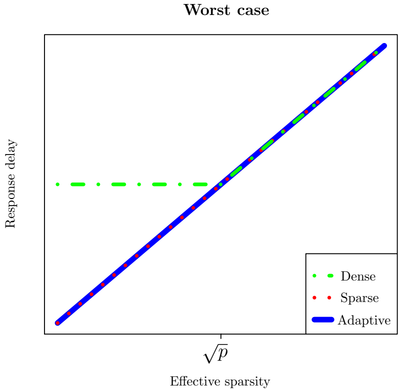
$$\bar{\mathbb{E}}_\theta(N) \lesssim \left(\frac{\sqrt{p} \log(ep\gamma)}{\vartheta^2} \vee \frac{\sqrt{s} \log(ep/\beta) \log(ep)}{\beta^2} \right) \wedge \frac{s \log(ep\gamma) \log(ep)}{\beta^2},$$

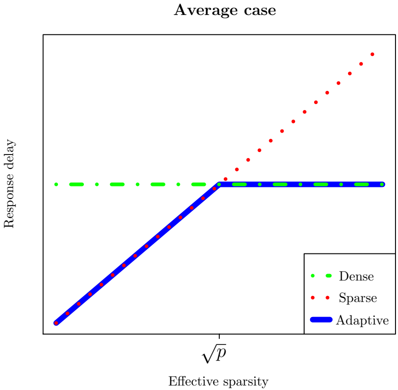
for all sufficiently small $\beta < \beta_0(s)$.

Response delays vs. sparsity

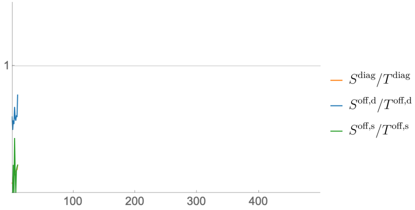
Assume that $\vartheta \asymp \beta \lesssim 1$ and $\log(\gamma/\beta) \lesssim \log p$. Then

$$\bar{\mathbb{E}}_\theta^{\mathrm{wc}}(N) \lesssim \frac{s \log^2(ep)}{\vartheta^2} \quad \text{and} \quad \bar{\mathbb{E}}_\theta(N) \lesssim \frac{(s \wedge p^{1/2}) \log^2(ep)}{\vartheta^2}.$$

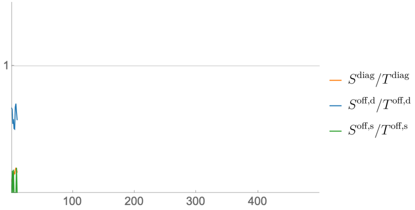




Setting: $p = 100, z = 900, \vartheta = \beta = 1, \gamma = 5000$



$s = 3$



$s = 100$

Comparison with other methods

We compare `ocd` with other recently proposed methods:

- `Mei`: ℓ_1 and ℓ_∞ aggregation of likelihood ratio tests in each coordinate. (Mei, 2010)
- `XS`: Use window-based method to aggregate statistics for testing the null against a normal mixture in each coordinate. (Xie and Siegmund, 2013)
- `Chan`: Similar to `XS`, but with an improved choice of tuning parameters. (Chan, 2017)

Simulation settings: $p \in \{100, 2000\}$, $s \in \{5, \lfloor\sqrt{p}\rfloor, p\}$, $\vartheta \in \{1, 0.5, 0.25\}$ and θ generated as ϑU, where U is uniformly distributed on the union of all s sparse unit spheres in $\mathbb{R}^p$.

- All thresholds are determined using Monte Carlo simulation.

Comparison with other methods

p	s	ϑ	ocd	Mei	XS	Chan
100	5	1	46.9	125.9	47.3	**42.0**
100	5	0.5	174.8	383.1	194.3	**163.7**
100	5	0.25	**583.5**	970.4	2147	1888.8
100	10	1	53.8	150.1	52.9	**51.5**
100	10	0.5	**194.4**	458.2	255.8	245.6
100	10	0.25	**629.7**	1171.3	2730.7	2484.9
100	100	1	**74.4**	268.3	89.6	102.1
100	100	0.5	**287.9**	834.9	526.8	756.0
100	100	0.25	**1005.8**	1912.9	3598.3	3406.6
2000	5	1	67.3	316.7	79.5	**59.5**
2000	5	0.5	**247.3**	680.2	607.7	285.0
2000	5	0.25	**851.3**	1384.8	4459.2	3856.9
2000	44	1	**136.0**	596.1	149.1	145.0
2000	44	0.5	**479.1**	1270.8	2945.5	2751.4
2000	44	0.25	**1584.2**	2428.8	4457.8	5049.7
2000	2000	1	**360.7**	2126.5	1020.0	2074.7
2000	2000	0.5	**1296.0**	3428.1	4669.3	4672.7
2000	2000	0.25	**3436.7**	4140.4	5063.7	5233.5

Table: Estimated response delay for ocd, Mei, XS and Chan over 200 repetitions, with $z = 0$ and $\gamma = 5000$.

Summary

- We propose a new, multiscale method for high-dimensional online changepoint detection.
- We perform likelihood ratio tests against simple alternatives of different scales in each coordinate, and aggregate these statistics.
- **R** package **ocd** is available on CRAN.

Main reference

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