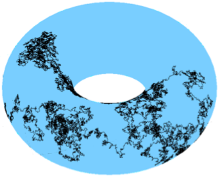
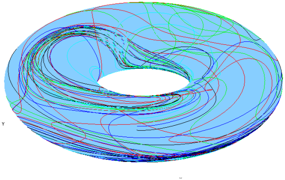
Kinetic Brownian motion in the diffeomorphism group of a closed Riemannian manifold





Joint works with J. Angst, C. Tardif and P. Perruchaud (Rennes)

1. Kinetic Brownian motion in $\mathbb{R}^d$

- Definition. **Kinetic Brownian motion** $(x_t, \dot{x}_t)$ **in** $\mathbb{R}^d$ is the hypoelliptic diffusion with state space $\mathbb{R}^d \times \mathbb{S}^{d-1}$

$$dx_t = \dot{x}_t \, dt,$$
$$\dot{x}_t = B_{\sigma^2 t},$$

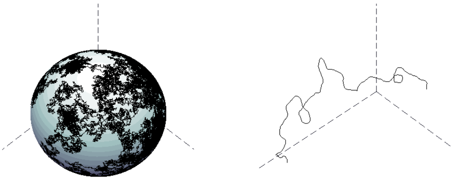
with B Brownian motion on $\mathbb{S}^{d-1}$, with parameter $\sigma \in [0, \infty)$.

1. Kinetic Brownian motion in $\mathbb{R}^d$

► Definition. **Kinetic Brownian motion** $(x_t, \dot{x}_t)$ **in** $\mathbb{R}^d$ is the hypoelliptic diffusion with state space $\mathbb{R}^d \times \mathbb{S}^{d-1}$

$$dx_t = \dot{x}_t \, dt,$$
$$\dot{x}_t = B_{\sigma^2 t},$$

with B Brownian motion on $\mathbb{S}^{d-1}$, with parameter $\sigma \in [0, \infty)$.

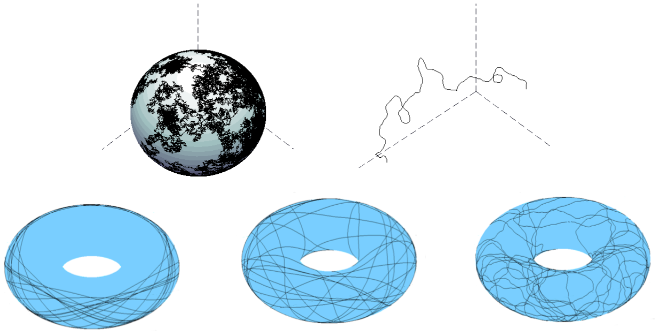


1. Kinetic Brownian motion in $\mathbb{R}^d$

► Definition. **Kinetic Brownian motion** $(x_t, \dot{x}_t)$ **in** $\mathbb{R}^d$ is the hypoelliptic diffusion with state space $\mathbb{R}^d \times \mathbb{S}^{d-1}$

$$dx_t = \dot{x}_t \, dt,$$
$$\dot{x}_t = B_{\sigma^2 t},$$

with B Brownian motion on $\mathbb{S}^{d-1}$, with parameter $\sigma \in [0, \infty)$.

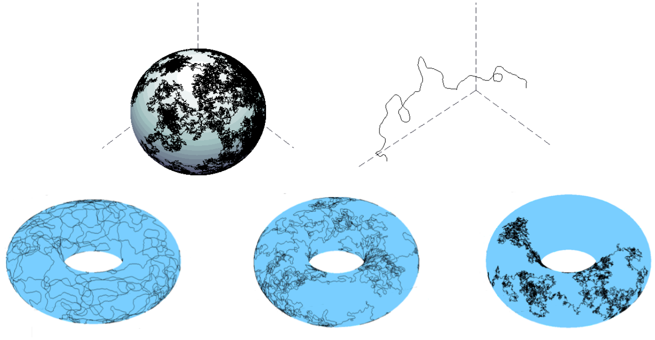


1. Kinetic Brownian motion in $\mathbb{R}^d$

- Definition. **Kinetic Brownian motion** $(x_t, \dot{x}_t)$ **in** $\mathbb{R}^d$ is the hypoelliptic diffusion with state space $\mathbb{R}^d \times \mathbb{S}^{d-1}$

$$
\begin{aligned}
dx_t &= \dot{x}_t \, dt, \\
\dot{x}_t &= B_{\sigma^2 t},
\end{aligned}
$$

with B Brownian motion on $\mathbb{S}^{d-1}$, with parameter $\sigma \in [0, \infty)$.



1. Kinetic Brownian motion in $\mathbb{R}^d$ – a hypoelliptic diffusion

- Even in this simple situation, no general result on heat kernel estimates is available.

Perruchaud proved in this PhD thesis an asymptotics in terms of the heat kernel $\overline{p}_t$ of a model *non-Gaussian* diffusion, with an explicit kernel, in a 2-dimensional setting

$$T^1\mathbb{T}^2 \simeq \mathbb{T}^2 \times \mathbb{S}^1 = \big\{(z,\theta)\big\} = \big\{((x,y),\theta)\big\}.$$

1. Kinetic Brownian motion in $\mathbb{R}^d$ – a hypoelliptic diffusion

• Even in this simple situation, no general result on heat kernel estimates is available.

Perruchaud proved in this PhD thesis an asymptotics in terms of the heat kernel $\overline{p}_t$ of a model *non-Gaussian* diffusion, with an explicit kernel, in a 2-dimensional setting

$$T^1\mathbb{T}^2 \simeq \mathbb{T}^2 \times \mathbb{S}^1 = \big\{(z,\theta)\big\} = \big\{((x,y),\theta)\big\}.$$

► Theorem (Perruchaud '19) – **Heat kernel estimate.** *Let D be a domain of $T^1\mathbb{T}^2$ where $\overline{p}_1\big((0,0),\bullet + ((1,0),0)\big)$ is bounded away from 0. Then*

$$p_t\big((0,0),((x+t,y),\theta)\big) = \overline{p}_t\big((0,0),((x+t,y),\theta)\big)\,(1+O(t)),$$

uniformly in $\left(t^{-2}x, t^{-3/2}y, t^{-1/2}\theta\right) \in D$.

1. Kinetic Brownian motion in $\mathbb{R}^d$ – a hypoelliptic diffusion

• Even in this simple situation, no general result on heat kernel estimates is available.

Perruchaud proved in this PhD thesis an asymptotics in terms of the heat kernel $\overline{p}_t$ of a model *non-Gaussian* diffusion, with an explicit kernel, in a 2-dimensional setting

$$T^1\mathbb{T}^2 \simeq \mathbb{T}^2 \times \mathbb{S}^1 = \big\{(z,\theta)\big\} = \big\{((x,y),\theta)\big\}.$$

► Theorem (Perruchaud '19) – **Heat kernel estimate.** *Let D be a domain of $T^1\mathbb{T}^2$ where $\overline{p}_1\big((0,0),\bullet + ((1,0),0)\big)$ is bounded away from 0. Then*

$$p_t\big((0,0),((x+t,y),\theta)\big) = \overline{p}_t\big((0,0),((x+t,y),\theta)\big)\,(1+O(t)),$$

uniformly in $\big(t^{-2}x, t^{-3/2}y, t^{-1/2}\theta\big) \in D$.

• A. Drouot 17' and H.F. Smith 20' provide hypoelliptic regularity estimates and a parametrix for the generator of kinetic Brownian motion.

1. Kinetic Brownian motion in $\mathbb{R}^d$

► Theorem – **Homogenization.** *The time-rescaled position process* $(x_{\sigma^2 t})_{0\leq t\leq 1}$ *converges weakly to a Euclidean Brownian motion with generator* $\frac{4}{d(d-1)}\Delta$*, as* $\sigma \uparrow \infty$.

1. Kinetic Brownian motion in $\mathbb{R}^d$

► Theorem – **Homogenization.** *The time-rescaled position process* $(x_{\sigma^2 t})_{0\le t\le 1}$ *converges weakly to a Euclidean Brownian motion with generator* $\frac{4}{d(d-1)}\Delta$*, as* $\sigma \uparrow \infty$*.*

Idea of proof. The dynamics of kinetic Brownian motion is given by the SDE

$$
\begin{aligned}
dx_t^i &= \dot{x}_t^i \, dt \\
d\dot{x}_t^i &= -\sigma^2 \frac{d-1}{2} \, \dot{x}_t^i dt + \sigma \sum_{j=1}^{d} \left(\delta^{ij} - \dot{x}_t^i \dot{x}_t^j\right) dW_t^j
\end{aligned}
$$

1. Kinetic Brownian motion in $\mathbb{R}^d$

▶ Theorem – **Homogenization.** *The time-rescaled position process* $(x_{\sigma^2 t})_{0\leq t\leq 1}$ *converges weakly to a Euclidean Brownian motion with generator* $\frac{4}{d(d-1)}\,\Delta$, *as* $\sigma \uparrow \infty$.

Idea of proof. The dynamics of kinetic Brownian motion is given by the SDE

$$
\begin{aligned}
dx_t^i &= \dot{x}_t^i \, dt \\
d\dot{x}_t^i &= -\sigma^2 \frac{d-1}{2}\, \dot{x}_t^i dt + \sigma \sum_{j=1}^{d} \left(\delta^{ij} - \dot{x}_t^i \dot{x}_t^j\right) dW_t^j
\end{aligned}
$$

Set $X_t^\sigma := x_{\sigma^2 t}$. Then

$$
X_t^\sigma = x_0 + \frac{2}{d-1}\,\frac{1}{\sigma^2}\left(\dot{x}_0 - \dot{x}_{\sigma^2 t}\right) + M_t^\sigma,
$$

with

$$
\left\langle M^{\sigma,i}, M^{\sigma,j}\right\rangle_t = \frac{4}{(d-1)^2}\,\frac{1}{\sigma^2}\int_0^{\sigma^2 t}\left(\delta^{ij} - \dot{x}_s^{\sigma,i}\dot{x}_s^{\sigma,j}\right) ds.
$$

1. Kinetic Brownian motion in $\mathbb{R}^d$

▶ Theorem – **Homogenization.** *The time-rescaled position process* $(x_{\sigma^2 t})_{0\leq t\leq 1}$ *converges weakly to a Euclidean Brownian motion with generator* $\frac{4}{d(d-1)}\Delta$, *as* $\sigma \uparrow \infty$.

Idea of proof. The dynamics of kinetic Brownian motion is given by the SDE

$$
\begin{aligned}
dx_t^i &= \dot{x}_t^i \, dt \\
d\dot{x}_t^i &= -\sigma^2 \frac{d-1}{2} \, \dot{x}_t^i dt + \sigma \sum_{j=1}^{d} \left(\delta^{ij} - \dot{x}_t^i \dot{x}_t^j\right) dW_t^j
\end{aligned}
$$

Set $X_t^\sigma := x_{\sigma^2 t}$. Then

$$
X_t^\sigma = x_0 + \frac{2}{d-1} \frac{1}{\sigma^2} \left(\dot{x}_0 - \dot{x}_{\sigma^2 t}\right) + M_t^\sigma,
$$

with

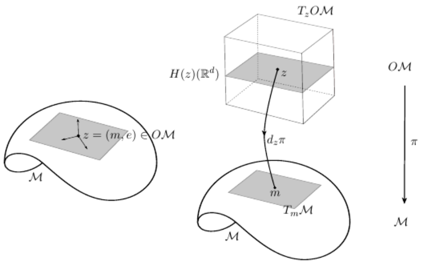
$$
\left\langle M^{\sigma,i}, M^{\sigma,j} \right\rangle_t = \frac{4}{(d-1)^2} \frac{1}{\sigma^2} \int_0^{\sigma^2 t} \left(\delta^{ij} - \dot{x}_s^{\sigma,i} \dot{x}_s^{\sigma,j}\right) ds.
$$

Use **ergodic theorem** and **functional CLT** to conclude. ◁

2. Manifold-valued Kinetic Brownian motion

Let (M, g) be a d-dimensional Riemannian manifold.

- **Cartan development**: a useful way to construct Brownian motion on M. Let $\pi : OM \to M$, stand for the **orthonormal frame bundle** over M; generic point $z = (m, e)$, with e orthonormal basis of T_mM. For $z \in OM$, let $H(z) \in L(\mathbb{R}^d, T_zOM)$ stand for the (metric-dependent) horizontal form at z.

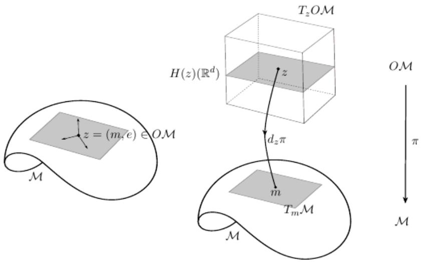


$dz_t = H(z_t) \circ dW_t$, in OM, $\qquad m_t := \pi(z_t)$, in M.

2. Manifold-valued Kinetic Brownian motion

Let (M, g) be a d-dimensional Riemannian manifold.

- **Cartan development**: a useful way to construct Brownian motion on M. Let $\pi : OM \to M$, stand for the **orthonormal frame bundle** over M; generic point $z = (m, e)$, with e orthonormal basis of $T_m M$. For $z \in OM$, let $H(z) \in L(\mathbb{R}^d, T_z OM)$ stand for the (metric-dependent) horizontal form at z.



$$dz_t = H(z_t) \circ dW_t, \text{ in OM}, \qquad m_t := \pi(z_t), \text{ in M}.$$

- Definition. **Kinetic Brownian motion m_t^σ in M via Cartan development.** For X_t^σ time rescaled kinetic Brownian motion in $\mathbb{R}^d$, set

$$dz_t^\sigma = H(z_t^\sigma)\, \dot{X}_t^\sigma\, dt, \text{ in OM}, \qquad m_t^\sigma := \pi(z_t^\sigma), \text{ in M}.$$

A parenthesis: Controlled ODEs and rough paths

$$dz_t = V(z_t)dX_t \qquad \big(X_t \in \mathbb{R}^\ell,\ z \in M,\ V(z) \in L(\mathbb{R}^\ell, T_z M)\big)$$

A parenthesis: Controlled ODEs and rough paths

$$dz_t = V(z_t)dX_t \qquad \big(X_t \in \mathbb{R}^\ell,\ z \in M,\ V(z) \in L(\mathbb{R}^\ell, T_z M)\big)$$

Robust theory of controlled ODEs for drivers $X \in W^{1,1}$, or $X \in C^\alpha$, for $\alpha > 1/2$, using Young integral formulation: Solution path $z \in C^0$ is a continuous function of control X.

A parenthesis: Controlled ODEs and rough paths

$$dz_t = V(z_t)dX_t \qquad \big(X_t \in \mathbb{R}^\ell,\ z \in M,\ V(z) \in L(\mathbb{R}^\ell, T_zM)\big)$$

Robust theory of controlled ODEs for drivers $X \in W^{1,1}$, or $X \in C^\alpha$, for $\alpha > 1/2$, using Young integral formulation: Solution path $z \in C^0$ is a continuous function of control X. One needs **rough paths** theory to handle *rougher controls*.

- A non-linear notion of control = rough paths, elements of a metric space.
- A notion of integral against a rough path.

A parenthesis: Controlled ODEs and rough paths

$$dz_t = V(z_t)dX_t \qquad \left(X_t \in \mathbb{R}^\ell,\, z \in M,\, V(z) \in L(\mathbb{R}^\ell, T_zM)\right)$$

Robust theory of controlled ODEs for drivers $X \in W^{1,1}$, or $X \in C^\alpha$, for $\alpha > 1/2$, using Young integral formulation: Solution path $z \in C^0$ is a continuous function of control X. One needs **rough paths** theory to handle *rougher controls*.

- A non-linear notion of control = rough paths, elements of a metric space.
- A notion of integral against a rough path.

One gets a robust theory of rough differential equations where

Solution path $z \in C^0$ is a continuous function of control X

A parenthesis: Controlled ODEs and rough paths

$$dz_t = V(z_t)dX_t \qquad \left(X_t \in \mathbb{R}^\ell,\ z \in M,\ V(z) \in L(\mathbb{R}^\ell, T_z M)\right)$$

Robust theory of controlled ODEs for drivers $X \in W^{1,1}$, or $X \in C^\alpha$, for $\alpha > 1/2$, using Young integral formulation: Solution path $z \in C^0$ is a continuous function of control X. One needs **rough paths** theory to handle *rougher controls*.

- A non-linear notion of control = rough paths, elements of a metric space.
- A notion of integral against a rough path.

One gets a robust theory of rough differential equations where

Solution path $z \in C^0$ is a continuous function of control X

A rough path is an abstract analogue of a tuple of iterated integrals

$$X_t - X_s, \left(\int_{s\le s_1\le s_2\le t} dX^{i_2}_{s_2}\, dX^{i_1}_{s_1}\right)_{1\le i_1,i_2\le\ell}, \left(\int_{s\le s_1\le s_2\le s_3\le t} dX^{i_3}_{s_3}\, dX^{i_2}_{s_2}\, dX^{i_1}_{s_1}\right)_{1\le i_1,i_2,i_3\le\ell}, \text{ etc.}$$

(These iterated integrals do not make sense when X is α-Hölder with $\alpha \le 1/2$.)

A parenthesis: Controlled ODEs and rough paths

$$dz_t = V(z_t)dX_t \qquad \big(X_t \in \mathbb{R}^\ell,\ z \in M,\ V(z) \in L(\mathbb{R}^\ell, T_zM)\big)$$

Robust theory of controlled ODEs for drivers $X \in W^{1,1}$, or $X \in C^\alpha$, for $\alpha > 1/2$, using Young integral formulation: Solution path $z \in C^0$ is a continuous function of control X. One needs **rough paths** theory to handle *rougher controls*.

- A non-linear notion of control = rough paths, elements of a metric space.
- A notion of integral against a rough path.

One gets a robust theory of rough differential equations where

Solution path $z \in C^0$ is a continuous function of control X

A rough path is an abstract analogue of a tuple of iterated integrals

$$X_t - X_s, \left(\int_{s \le s_1 \le s_2 \le t} dX^{i_2}_{s_2}\, dX^{i_1}_{s_1}\right)_{1 \le i_1, i_2 \le \ell}, \left(\int_{s \le s_1 \le s_2 \le s_3 \le t} dX^{i_3}_{s_3}\, dX^{i_2}_{s_2}\, dX^{i_1}_{s_1}\right)_{1 \le i_1, i_2, i_3 \le \ell}, \textit{etc.}$$

(These iterated integrals do not make sense when X is α-Hölder with $\alpha \le 1/2$.)

Definition involves

- a multi-level object indexed by $(0 \le s \le t)$,
- algebraic constraints between its components,
- analytic constraints on the size of its components as functions of (s, t).

A parenthesis: Controlled ODEs and rough paths

$$dz_t = V(z_t)dX_t \qquad \big(X_t \in \mathbb{R}^\ell,\ z \in M,\ V(z) \in L(\mathbb{R}^\ell, T_z M)\big)$$

Robust theory of controlled ODEs for drivers $X \in W^{1,1}$, or $X \in C^\alpha$, for $\alpha > 1/2$, using Young integral formulation: Solution path $z \in C^0$ is a continuous function of control X. One needs **rough paths** theory to handle *rougher controls*.

- A non-linear notion of control = rough paths, elements of a metric space.
- A notion of integral against a rough path.

One gets a robust theory of rough differential equations where

Solution path $z \in C^0$ is a continuous function of control X

Continuity of the solution map $X \mapsto z$ to a rough differential equation allows to transport support theorems, large deviation theorems, weak convergence results for random rough paths to the random solution paths.

A parenthesis: Controlled ODEs and rough paths

$$dz_t = V(z_t)dX_t \qquad \big(X_t \in \mathbb{R}^\ell,\ z \in M,\ V(z) \in L(\mathbb{R}^\ell, T_zM)\big)$$

Robust theory of controlled ODEs for drivers $X \in W^{1,1}$, or $X \in C^\alpha$, for $\alpha > 1/2$, using Young integral formulation: Solution path $z \in C^0$ is a continuous function of control X. One needs **rough paths** theory to handle *rougher controls*.

- A non-linear notion of control = rough paths, elements of a metric space.
- A notion of integral against a rough path.

One gets a robust theory of rough differential equations where

Solution path $z \in C^0$ is a continuous function of control X

Continuity of the solution map $X \mapsto z$ to a rough differential equation allows to transport support theorems, large deviation theorems, weak convergence results for random rough paths to the random solution paths.

If B is a Brownian motion and $\mathbf{B} = (B, \mathbb{B})$ with

$$\mathbb{B}_{ts} := \int (B_u - B_s) \otimes \circ dB_u.$$

the solution to the rough differential equation

$$dz_t = V(z_t)d\mathbf{B}_t$$

coincides almost surely with the solution of the Stratonovich SDE

$$dz_t = V(z_t)\circ dB_t.$$

A parenthesis: Controlled ODEs and rough paths

$$dz_t = V(z_t)dX_t \qquad \big(X_t \in \mathbb{R}^\ell,\ z \in M,\ V(z) \in L(\mathbb{R}^\ell, T_z M)\big)$$

Robust theory of controlled ODEs for drivers $X \in W^{1,1}$, or $X \in C^\alpha$, for $\alpha > 1/2$, using Young integral formulation: Solution path $z \in C^0$ is a continuous function of control X. One needs **rough paths** theory to handle *rougher controls*.

- A non-linear notion of control = rough paths, elements of a metric space.
- A notion of integral against a rough path.

One gets a robust theory of rough differential equations where

Solution path $z \in C^0$ is a continuous function of control X

Continuity of the solution map $X \mapsto z$ to a rough differential equation allows to transport support theorems, large deviation theorems, weak convergence results for random rough paths to the random solution paths.

Back to kinetic Brownian motion on a Riemannian manifold M

$$dz_t^\sigma = H(z_t^\sigma)\, dX_t^\sigma, \text{ in } OM, \quad m_t^\sigma := \pi(z_t^\sigma), \text{ in } M,$$

with X_t^σ kinetic Brownian motion in $\mathbb{R}^d$.

2. Manifold-valued Kinetic Brownian motion

- Theorem (Bailleul-Angst-Tardif '15) – **Homogenization.** *Assume (M, g) is complete and stochastically complete. Then the process $\left(m_t^{\sigma}\right)_{0 \le t \le 1}$ converges weakly to a Brownian motion with generator $\frac{4}{d(d-1)}\Delta$, as $\sigma \uparrow \infty$.*

2. Manifold-valued Kinetic Brownian motion

► Theorem (Bailleul-Angst-Tardif '15) – **Homogenization.** *Assume (M, g) is complete and stochastically complete. Then the process $\big(m_t^\sigma\big)_{0\leq t\leq 1}$ converges weakly to a Brownian motion with generator $\frac{4}{d(d-1)}\Delta$, as $\sigma \uparrow \infty$.*

Idea of proof. Back in $\mathbb{R}^d$ with time rescaled kinetic Brownian motion X_t^σ. Prove that the canonical rough path lift $\mathbf{X}^\sigma$ of $(X_t^\sigma)_{0\leq t\leq 1}$ converges weakly in a rough path sense to the Stratonovich Brownian rough path.

◁

2. Manifold-valued Kinetic Brownian motion

► Theorem (Bailleul-Angst-Tardif '15) – **Homogenization.** *Assume (M, g) is complete and stochastically complete. Then the process $\big(m_t^\sigma\big)_{0\leq t\leq 1}$ converges weakly to a Brownian motion with generator $\frac{4}{d(d-1)}\Delta$, as $\sigma \uparrow \infty$.*

Idea of proof. Back in $\mathbb{R}^d$ with time rescaled kinetic Brownian motion X_t^σ. Prove that the canonical rough path lift $\mathbf{X}^\sigma$ of $(X_t^\sigma)_{0\leq t\leq 1}$ converges weakly in a rough path sense to the Stratonovich Brownian rough path.

Use the continuity of the Itô-Lyons solution map for the rough differential equation

$$dz_t^\sigma = H(z_t^\sigma)\, dX_t^\sigma = H(z_t^\sigma)\, d\mathbf{X}_t^\sigma, \quad z_t^\sigma \in OM,$$

to transport weak convergence of $\mathbf{X}^\sigma$ from the rough paths side to the dynamics on OM and M. ◁

2. Manifold-valued Kinetic Brownian motion

► Theorem (Bailleul-Angst-Tardif '15) – **Homogenization.** *Assume (M, g) is complete and stochastically complete. Then the process $\big(m^\sigma_t\big)_{0\leq t\leq 1}$ converges weakly to a Brownian motion with generator $\frac{4}{d(d-1)}\Delta$, as $\sigma\uparrow\infty$.*

Idea of proof. Back in $\mathbb{R}^d$ with time rescaled kinetic Brownian motion X^σ_t. Prove that the canonical rough path lift $\mathbf{X}^\sigma$ of $(X^\sigma_t)_{0\leq t\leq 1}$ converges weakly in a rough path sense to the Stratonovich Brownian rough path. Done as follows.

- Prove first weak convergence in uniform norm of $\mathbf{X}^\sigma$ to the Stratonovich Brownian rough path, using weak convergence results on stochastic integrals.
- Prove σ-uniform moment bounds on X^σ_{ts} and $\int_s^t X^\sigma_{us}\otimes dX^\sigma_u$, and use Lamperti-type tightness result for random rough paths.

Use the continuity of the Itô-Lyons solution map for the rough differential equation

$$dz^\sigma_t = H(z^\sigma_t)\, dX^\sigma_t = H(z^\sigma_t)\, d\mathbf{X}^\sigma_t, \quad z^\sigma_t \in OM,$$

to transport weak convergence of $\mathbf{X}^\sigma$ from the rough paths side to the dynamics on OM and M. ◁

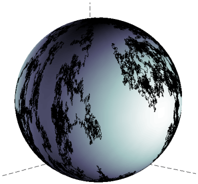
3. Anisotropic Kinetic Brownian motion in $\mathbb{R}^d$

Let Σ be a positive-definite symmetric matrix – no loss in assuming $\Sigma = \mathrm{diag}(\alpha_i^2)$.

► Definition. **Anisotropic Kinetic Brownian motion** $(x_t, \dot{x}_t)$ **in** $\mathbb{R}^d$, with anisotropy Σ, is the hypoelliptic diffusion with state space $\mathbb{R}^d \times \mathbb{S}^{d-1}$

$$dx_t = \dot{x}_t \, dt,$$
$$d\dot{x}_t = \sigma P_{\dot{x}_t} \circ dW_t,$$

where W is an $\mathbb{R}^d$-valued Brownian motion with covariance Σ, and $P_{\dot{x}} : \mathbb{R}^d \to \langle \dot{x} \rangle^\perp$, the orthogonal projection. (Note $\langle \dot{x} \rangle^\perp = T_{\dot{x}} \mathbb{S}^{d-1}$.)



3. Anisotropic Kinetic Brownian motion on $\mathbb{R}^d$

- Theorem (Perruchaud '19) – **Homogenization.**
  - *The invariant measure μ of the velocity process $\dot{x}$ on the sphere is the image by the radial projection on the sphere of the measure on $\mathbb{R}^d$ with density $|x|^{-1}$ wrt the Gaussian measure with covariance Σ.*

3. Anisotropic Kinetic Brownian motion on $\mathbb{R}^d$

- Theorem (Perruchaud '19) – **Homogenization.**
  - *The invariant measure* μ *of the velocity process* $\dot{x}$ *on the sphere is the image by the radial projection on the sphere of the measure on* $\mathbb{R}^d$ *with density* $|x|^{-1}$ *wrt the Gaussian measure with covariance* Σ*.*
  - *The time-rescaled process* $(x_{\sigma^2 t})_{0\leq t\leq 1}$ *converges weakly as* $\sigma \uparrow \infty$ *to a Euclidean Brownian motion with covariance matrix* $\mathrm{diag}(\gamma_i)$*, with*

$$\gamma_i := 2\int_0^\infty \mathbb{E}_\mu\left[\dot{x}_0^i\,\dot{x}_t^i\right] dt, \quad 1 \leq i \leq d.$$

3. Anisotropic Kinetic Brownian motion on $\mathbb{R}^d$

- Theorem (Perruchaud '19) – **Homogenization.**
  - *The invariant measure μ of the velocity process $\dot{x}$ on the sphere is the image by the radial projection on the sphere of the measure on $\mathbb{R}^d$ with density $|x|^{-1}$ wrt the Gaussian measure with covariance Σ.*
  - *The time-rescaled process $(x_{\sigma^2 t})_{0 \leq t \leq 1}$ converges weakly as $\sigma \uparrow \infty$ to a Euclidean Brownian motion with covariance matrix* $\mathrm{diag}(\gamma_i)$, *with*

$$\gamma_i := 2 \int_0^\infty \mathbb{E}_\mu\left[\dot{x}_0^i \, \dot{x}_t^i\right] dt, \quad 1 \leq i \leq d.$$

  - We have weak convergence of the associated rough path $\mathbf{X}^\sigma$ to the corresponding Stratonovich Brownian rough path, as $\sigma \uparrow \infty$.

3. Anisotropic Kinetic Brownian motion on $\mathbb{R}^d$

► Theorem (Perruchaud '19) – **Homogenization.**

- *The invariant measure* μ *of the velocity process* $\dot{x}$ *on the sphere is the image by the radial projection on the sphere of the measure on* $\mathbb{R}^d$ *with density* $|x|^{-1}$ *wrt the Gaussian measure with covariance* Σ*.*
- *The time-rescaled process* $(x_{\sigma^2 t})_{0\leq t\leq 1}$ *converges weakly as* $\sigma \uparrow \infty$ *to a Euclidean Brownian motion with covariance matrix* $\mathrm{diag}(\gamma_i)$*, with*

$$\gamma_i := 2\int_0^\infty \mathbb{E}_\mu\big[\dot{x}_0^i\,\dot{x}_t^i\big]\,dt, \quad 1\leq i\leq d.$$

- We have weak convergence of the associated rough path $\mathbf{X}^\sigma$ to the corresponding Stratonovich Brownian rough path, as $\sigma\uparrow\infty$.

Idea of proof. The dynamics of velocity $\dot{x}_t$ is given by the SDE

$$d\dot{x}_t^i = -\frac{\sigma^2}{2}\left(\alpha_i^2 + \sum_{k=1}^d \alpha_k^2 - 2\sum_{\ell=1}^d \alpha_\ell^2\,\big|\dot{x}_t^\ell\big|^2\right)\dot{x}_t^i\,dt + \sigma\left(\alpha_i dW_t^i - \dot{x}_t^i\sum_{\ell=1}^d \alpha_\ell \dot{x}_t^\ell dW_t^\ell\right)$$

No clear description of $X_t^\sigma = x_{\sigma^2 t}$, when Σ different from a constant multiple of identity. Give up the analysis of the SDE and **use ergodic properties of** $\dot{x}$.

3. Anisotropic Kinetic Brownian motion on $\mathbb{R}^d$

1. One has for any probability measure λ on $\mathbb{S}^{d-1}$

$$\left\| P_t^* \lambda - \mu \right\|_{\mathrm{TV}} \lesssim e^{-ct},$$

for some positive constant c.

3. Anisotropic Kinetic Brownian motion on $\mathbb{R}^d$

1. One has for any probability measure λ on $\mathbb{S}^{d-1}$

$$\left\| P_t^* \lambda - \mu \right\|_{\mathrm{TV}} \lesssim e^{-ct},$$

for some positive constant c. This implies σ-uniform moment estimates

$$\sup_{\sigma \geq 0} \left\| X_t^\sigma - X_s^\sigma \right\|_{L^p} \lesssim |t-s|^{p/2},$$
$$\sup_{\sigma \geq 0} \left\| \mathbb{X}_{ts}^\sigma \right\|_{L^p} \lesssim |t-s|^{p},$$

where

$$\mathbb{X}_{ts}^\sigma := \int_s^t X_{us}^\sigma \otimes dX_u^\sigma,$$

implying **tightness** for the laws of the canonical rough paths $\mathbf{X}^\sigma$ associated with anisotropic kinetic Brownian motion.

3. Anisotropic Kinetic Brownian motion on $\mathbb{R}^d$

1. One has for any probability measure λ on $\mathbb{S}^{d-1}$

$$\left\| P_t^* \lambda - \mu \right\|_{\mathrm{TV}} \lesssim e^{-ct},$$

for some positive constant c. This implies σ-uniform moment estimates

$$\sup_{\sigma \geq 0} \left\| X_t^\sigma - X_s^\sigma \right\|_{L^p} \lesssim |t-s|^{p/2},$$

$$\sup_{\sigma \geq 0} \left\| \mathbb{X}_{ts}^\sigma \right\|_{L^p} \lesssim |t-s|^{p},$$

where

$$\mathbb{X}_{ts}^\sigma := \int_s^t X_{us}^\sigma \otimes dX_u^\sigma,$$

implying **tightness** for the laws of the canonical rough paths $\mathbf{X}^\sigma$ associated with anisotropic kinetic Brownian motion.

2. One proves that any limit law turns the canonical process on the rough paths space into a continuous **Lévy process**.

3. Anisotropic Kinetic Brownian motion on $\mathbb{R}^d$

1. One has for any probability measure λ on $\mathbb{S}^{d-1}$

$$\left\| P_t^* \lambda - \mu \right\|_{\mathrm{TV}} \lesssim e^{-ct},$$

for some positive constant c. This implies σ-uniform moment estimates

$$\sup_{\sigma \geq 0} \left\| X_t^\sigma - X_s^\sigma \right\|_{L^p} \lesssim |t-s|^{p/2},$$
$$\sup_{\sigma \geq 0} \left\| \mathbb{X}_{ts}^\sigma \right\|_{L^p} \lesssim |t-s|^{p},$$

where

$$\mathbb{X}_{ts}^\sigma := \int_s^t X_{us}^\sigma \otimes dX_u^\sigma,$$

implying **tightness** for the laws of the canonical rough paths $\mathbf{X}^\sigma$ associated with anisotropic kinetic Brownian motion.

2. One proves that any limit law turns the canonical process on the rough paths space into a continuous **Lévy process**. We identify its generator using the invariance of the invariant measure μ by the **symmetries**

$$(\theta_1, \dots, \theta_d) \in \mathbb{S}^{d-1} \mapsto (\theta_1, \dots, \theta_{i-1}, -\theta_i, \theta_{i+1}, \dots, \theta_d) \in \mathbb{S}^{d-1}. \qquad \triangleleft$$

4. Geometry of the diffeomorphism group

- (M, g) a Riemannian manifold = **domain of the fluid flow**,

  $\mathscr{D} := \{\text{Diffeo of } M\}$ or $H^s(M, M)$: a Fréchet/Hilbert manifold,

  $T_\varphi \mathscr{D} = \{\text{smooth}/H^s \text{ 'vector fields' at } \varphi\} = \{m \in M \to u(m) \in T_{\varphi(m)} M\}$.

(Variant with volume preserving diffeomorphism group and divergence-free vector fields on M.)

4. Geometry of the diffeomorphism group

- (M, g) a Riemannian manifold = **domain of the fluid flow**,
  $\mathscr{D} := \{\text{Diffeo of } M\}$ or $H^s(M, M)$: a Fréchet/Hilbert manifold,
  $T_\varphi \mathscr{D} = \{\text{smooth}/H^s \text{ 'vector fields' at } \varphi\} = \{m \in M \to u(m) \in T_{\varphi(m)} M\}$.

(Variant with volume preserving diffeomorphism group and divergence-free vector fields on M.)

- **Weak Riemannian metric** on $\mathscr{D}$

$$\langle u, v \rangle := \int_M g_{\varphi(m)}\big(u(m), v(m)\big)\, \mathrm{VOL}_g(dm).$$

Induced topology on $\mathscr{D}$ weaker than smooth or H^s topology. There may be no good notion of parallel transport... But Ebin-Marsden (69') prove there is one!
It is a *smooth map*, and its *exponential map* is *well-defined and smooth* in a neighbourhood of the zero section of $T\mathscr{D}$: $\mathrm{Exp}_{\mathrm{Id}}(u)(x) = \exp_x\big(u(x)\big)$.

4. Geometry of the diffeomorphism group

- (M, g) a Riemannian manifold = **domain of the fluid flow**,

  $\mathscr{D} := \{\text{Diffeo of } M\}$ or $H^s(M, M)$: a Fréchet/Hilbert manifold,

  $T_\varphi \mathscr{D} = \{\text{smooth}/H^s \text{ 'vector fields' at } \varphi\} = \{m \in M \to u(m) \in T_{\varphi(m)} M\}$.

(Variant with volume preserving diffeomorphism group and divergence-free vector fields on M.)

- **Weak Riemannian metric** on $\mathscr{D}$

$$\langle u, v \rangle := \int_M g_{\varphi(m)}\big(u(m), v(m)\big)\,\mathrm{VOL}_g(dm).$$

Induced topology on $\mathscr{D}$ weaker than smooth or H^s topology. There may be no good notion of parallel transport... But Ebin-Marsden (69') prove there is one!
It is a *smooth map*, and its *exponential map* is *well-defined and smooth* in a neighbourhood of the zero section of $T\mathscr{D}$: $\mathrm{Exp}_{\mathrm{Id}}(u)(x) = \exp_x\big(u(x)\big)$.

Geodesics (φ_t) on the 'submanifold' of volume preserving diffeomorphisms whose velocity fields $u = \partial_t \varphi_t \circ \varphi_t^{-1}$ are solutions of **Euler's equation for incompressible fluids**

$$\partial_t u + u \nabla u + \nabla p = 0,$$

for a pressure field $p : M \to \mathbb{R}$ ensuring that u remains divergence free.
(V.I. Arnol'd, 66')

4. Geometry of the diffeomorphism group

Shnirelman proved a number of stricking results.

- *The group of volume-preserving diffeomorphisms of the unit cube of $\mathbb{R}^n$ has infinite diameter in dimension 2, and infinite diameter in dimension $n \geq 3$.*

4. Geometry of the diffeomorphism group

Shnirelman proved a number of stricking results.

- *The group of volume-preserving diffeomorphisms of the unit cube of* $\mathbb{R}^n$ *has infinite diameter in dimension 2, and infinite diameter in dimension* $n \geq 3$*.*

A volume-preserving diffeomorphism f of M is said to be *attainable* if there exists a piecewise smooth path of volume-preserving diffeomorphisms with finite length from the identity to f. It is said to be *unattainable* otherwise. Work on the unit cube of $\mathbb{R}^n$.

- *In dimension* $n \geq 3$*, every volume-preserving diffeomorphism is attainable. In dimension* 2 *there exists unattainable diffeomorphisms.*

4. Geometry of the diffeomorphism group

Shnirelman proved a number of stricking results.

- *The group of volume-preserving diffeomorphisms of the unit cube of* $\mathbb{R}^n$ *has infinite diameter in dimension 2, and infinite diameter in dimension* $n \geq 3$.

A volume-preserving diffeomorphism f of M is said to be *attainable* if there exists a piecewise smooth path of volume-preserving diffeomorphisms with finite length from the identity to f. It is said to be *unattainable* otherwise. Work on the unit cube of $\mathbb{R}^n$.

- *In dimension* $n \geq 3$*, every volume-preserving diffeomorphism is attainable. In dimension* 2 *there exists unattainable diffeomorphisms.*

Pick $n \geq 3$.

- $\mathrm{dist}_{\mathscr{D}}(f,g) \lesssim \|d_M(f,g)\|_{L^2(M)}^{2/(n+4)}$ – *wrong in dimension* 2.

4. Geometry of the diffeomorphism group

Shnirelman proved a number of stricking results.

- *The group of volume-preserving diffeomorphisms of the unit cube of $\mathbb{R}^n$ has infinite diameter in dimension 2, and infinite diameter in dimension $n \geq 3$.*

A volume-preserving diffeomorphism f of M is said to be *attainable* if there exists a piecewise smooth path of volume-preserving diffeomorphisms with finite length from the identity to f. It is said to be *unattainable* otherwise. Work on the unit cube of $\mathbb{R}^n$.

- *In dimension $n \geq 3$, every volume-preserving diffeomorphism is attainable. In dimension 2 there exists unattainable diffeomorphisms.*

Pick $n \geq 3$.

- $\mathrm{dist}_{\mathscr{D}}(f, g) \lesssim \left\| d_M(f, g) \right\|_{L^2(M)}^{2/(n+4)}$ *– wrong in dimension 2.*

- *The $\mathrm{dist}_{\mathscr{D}}$-completion of the set of volume-preserving diffeomorphism group of the unit cube of $\mathbb{R}^n$ is the set of volume preserving measurable maps from M into itself.*

4. Geometry of the diffeomorphism group

Shnirelman proved a number of stricking results.

- *The group of volume-preserving diffeomorphisms of the unit cube of $\mathbb{R}^n$ has infinite diameter in dimension 2, and infinite diameter in dimension $n \geq 3$.*

A volume-preserving diffeomorphism f of M is said to be *attainable* if there exists a piecewise smooth path of volume-preserving diffeomorphisms with finite length from the identity to f. It is said to be *unattainable* otherwise. Work on the unit cube of $\mathbb{R}^n$.

- *In dimension $n \geq 3$, every volume-preserving diffeomorphism is attainable. In dimension 2 there exists unattainable diffeomorphisms.*

Pick $n \geq 3$.

- $\mathrm{dist}_{\mathscr{D}}(f, g) \lesssim \left\| d_M(f, g) \right\|_{L^2(M)}^{2/(n+4)}$ – *wrong in dimension 2.*

- *The $\mathrm{dist}_{\mathscr{D}}$-completion of the set of volume-preserving diffeomorphism group of the unit cube of $\mathbb{R}^n$ is the set of volume preserving measurable maps from M into itself.*

- *There exists pairs of volume-preserving diffeomorphisms of the cube that cannot be connected by a shortest path within the set of volume-preserving diffeomorphisms.*

4. Geometry of the diffeomorphism group

For the group of volume-preserving diffeomorphisms of the 2-dimensional torus $\mathbf{T}^2$:

- Orthonormal basis of the set $\textsc{Lie}(\mathscr{D})$ of null divergence vector fields. For $k \in \mathbb{Z}\backslash\{0\}$

$$A_k = |k|^{-1}\big(k_2 \cos(k \cdot \theta)\partial_1 - k_1 \cos(k \cdot \theta)\partial_2\big),$$
$$B_k = |k|^{-1}\big(k_2 \sin(k \cdot \theta)\partial_1 - k_1 \sin(k \cdot \theta)\partial_2\big).$$

4. Geometry of the diffeomorphism group

For the group of volume-preserving diffeomorphisms of the 2-dimensional torus $\mathbf{T}^2$:

- Orthonormal basis of the set $\mathrm{Lie}(\mathscr{D})$ of null divergence vector fields. For $k \in \mathbb{Z}\backslash\{0\}$

$$A_k = |k|^{-1}\big(k_2 \cos(k\cdot\theta)\partial_1 - k_1 \cos(k\cdot\theta)\partial_2\big),$$
$$B_k = |k|^{-1}\big(k_2 \sin(k\cdot\theta)\partial_1 - k_1 \sin(k\cdot\theta)\partial_2\big).$$

- Geodesic equation $u := \partial_t\varphi \circ \varphi^{-1}$

$$\partial_t u + \Gamma(u,u) = 0,$$

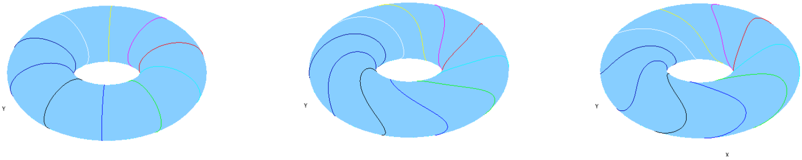
with explicit Christoffel symbols Γ, e.g.

$$\Gamma(A_k, A_\ell) = [k,\ell]\big(\alpha_{k,\ell}B_{k+\ell} + \beta_{k,\ell}B_{k-\ell}\big).$$

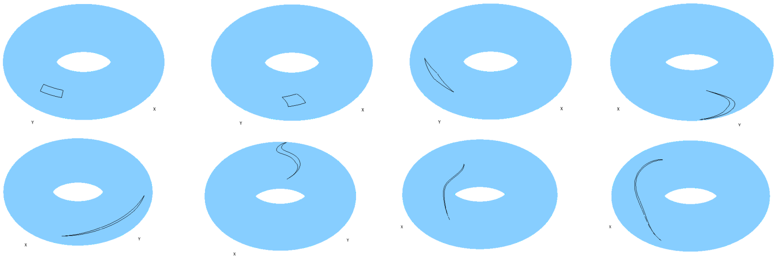
$\Gamma(A_k,\cdot), \Gamma(B_k,\cdot)$ unbounded antisymmetric operators that do not induce nice evolutions on the "orthonormal group" in $\mathrm{Lie}(\mathscr{D})$.

4. Geometry of the diffeomorphism group

- **Time 1 flow with $\sigma = 0$, for different initial momentum in volume preserving diffeomorphism group.**



- **Evolution with time of an area element along geodesic motion in volume preserving diffeomorphism group.**



5. Kinetic Brownian motion in the diffeomorphism group

We follow Cartan's development strategy, defining first a 'flat' kinetic Brownian motion in the space of vector fields (= the tangent space to identity of the manifold/group of diffeomorphisms), and then developing it on the manifold of diffeomorphisms.

5. Kinetic Brownian motion in the diffeomorphism group

We follow Cartan's development strategy, defining first a 'flat' kinetic Brownian motion in the space of vector fields (= the tangent space to identity of the manifold/group of diffeomorphisms), and then developing it on the manifold of diffeomorphisms.

1. On $\text{LIE}(\mathscr{D}) \simeq H^s(TM)$. Write $\mathbb{S}$ for unit sphere of $H^s(TM)$,

$$\begin{aligned} du_t &= \dot{u}_t\, dt, & u \in H^s(TM),\\ d\dot{u}_t &= \sigma P_{\dot{u}_t} \circ dW_t, & \dot{u} \in \mathbb{S}, \end{aligned}$$

with W an $H^s(TM)$-valued (anisotropic!) Brownian motion – with trace-class covariance operator Σ. This is kinetic Brownian motion on $\text{LIE}(\mathscr{D})$.

5. Kinetic Brownian motion in the diffeomorphism group

We follow Cartan's development strategy, defining first a 'flat' kinetic Brownian motion in the space of vector fields (= the tangent space to identity of the manifold/group of diffeomorphisms), and then developing it on the manifold of diffeomorphisms.

1. On $\mathrm{LIE}(\mathscr{D}) \simeq H^s(TM)$. Write $\mathbb{S}$ for unit sphere of $H^s(TM)$,

$$\begin{aligned} du_t &= \dot{u}_t\, dt, & u \in H^s(TM), \\ d\dot{u}_t &= \sigma P_{\dot{u}_t} \circ dW_t, & \dot{u} \in \mathbb{S}, \end{aligned}$$

with W an $H^s(TM)$-valued (anisotropic!) Brownian motion – with trace-class covariance operator Σ. This is kinetic Brownian motion on $\mathrm{LIE}(\mathscr{D})$.

2 Follow Ebin-Marsden' strategy, showing one can formulate Cartan's development operation as solving nice ODE on the infinite-dimensional configuration space (= a substitue for the orthonormal frame bundle above $\mathscr{D}$)

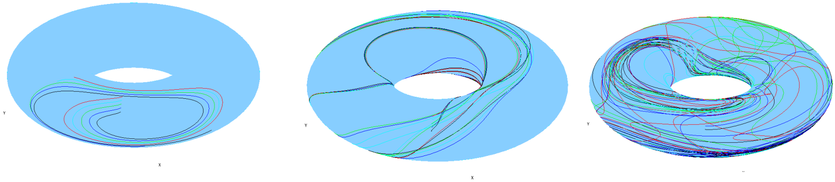
$$TH^s(\mathcal{F}M) \times L(H^s(TM)),$$

driven by a *smooth* vector field and controlled by u. Set $\varphi_t :=$ projection of dynamics on the diffeomorphism space $\mathscr{D}$.

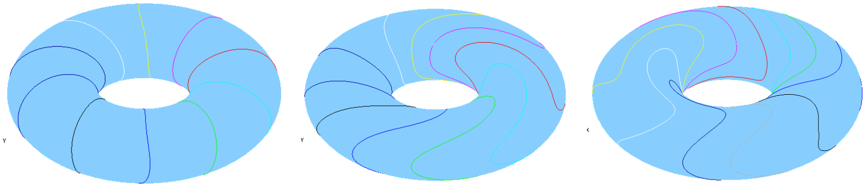
(Variant for volume-preserving diffeomorphism group and divergence-free vector fields on M.)

5. Kinetic Brownian motion in the diffeomorphism group

- **Examples of flows with time, for noise parameter $\sigma = 1$.**



- **Time 1 snapshots for increasing noise parameter σ, with same initial momentum.**



5. Kinetic Brownian motion in the diffeomorphism group

Set $U_t^\sigma := u_{\sigma^2 t} \in \mathrm{L{\scriptstyle IE}}(\mathscr{D})$. Wlog $\Sigma = \mathrm{diag}(\alpha_i^2)$, non-increasing eigenvalues α_i.

5. Kinetic Brownian motion in the diffeomorphism group

Set $U_t^\sigma := u_{\sigma^2 t} \in \mathrm{LIE}(\mathscr{D})$. Wlog $\Sigma = \mathrm{diag}(\alpha_i^2)$, non-increasing eigenvalues α_i.

► Theorem (Angst, Bailleul, Perruchaud 19') – **Homogenization in $\mathrm{LIE}(\mathscr{D})$.** Assume $3\,\alpha_1^2 < \mathrm{tr}(\Sigma)$ – there is sufficient noise in the system.

5. Kinetic Brownian motion in the diffeomorphism group

Set $U_t^\sigma := u_{\sigma^2 t} \in \mathrm{LIE}(\mathscr{D})$. Wlog $\Sigma = \mathrm{diag}(\alpha_i^2)$, non-increasing eigenvalues α_i.

► Theorem (Angst, Bailleul, Perruchaud 19') – **Homogenization in LIE($\mathscr{D}$).** Assume $3\,\alpha_1^2 < \mathrm{tr}(\Sigma)$ – there is sufficient noise in the system.

- *The velocity process in the unit sphere of* LIE($\mathscr{D}$) *is ergodic and converges exponentially fast to its unique invariant measure in* 2*-Wasserstein distance.*

5. Kinetic Brownian motion in the diffeomorphism group

Set $U_t^\sigma := u_{\sigma^2 t} \in \mathrm{LIE}(\mathscr{D})$. Wlog $\Sigma = \mathrm{diag}(\alpha_i^2)$, non-increasing eigenvalues α_i.

► Theorem (Angst, Bailleul, Perruchaud 19') – **Homogenization in $\mathrm{LIE}(\mathscr{D})$.** Assume $3\,\alpha_1^2 < \mathrm{tr}(\Sigma)$ – there is sufficient noise in the system.

- *The velocity process in the unit sphere of $\mathrm{LIE}(\mathscr{D})$ is ergodic and converges exponentially fast to its unique invariant measure in 2-Wasserstein distance.*
- *The invariant measure μ of the velocity process $\dot{u}$ on the unit sphere of $\mathrm{LIE}(\mathscr{D})$ is the image by the radial projection on the sphere of the measure on $\mathrm{LIE}(\mathscr{D})$ with density $|u|^{-1}$ wrt the Gaussian measure with covariance Σ.*

5. Kinetic Brownian motion in the diffeomorphism group

Set $U_t^\sigma := u_{\sigma^2 t} \in \mathrm{LIE}(\mathscr{D})$. Wlog $\Sigma = \mathrm{diag}(\alpha_i^2)$, non-increasing eigenvalues α_i.

► Theorem (Angst, Bailleul, Perruchaud 19') – **Homogenization in $\mathrm{LIE}(\mathscr{D})$.** Assume $3\,\alpha_1^2 < \mathrm{tr}(\Sigma)$ – there is sufficient noise in the system.

- *The velocity process in the unit sphere of $\mathrm{LIE}(\mathscr{D})$ is ergodic and converges exponentially fast to its unique invariant measure in 2-Wasserstein distance.*
- *The invariant measure μ of the velocity process $\dot{u}$ on the unit sphere of $\mathrm{LIE}(\mathscr{D})$ is the image by the radial projection on the sphere of the measure on $\mathrm{LIE}(\mathscr{D})$ with density $|u|^{-1}$ wrt the Gaussian measure with covariance Σ.*
- *The time-rescaled process $(U_t^\sigma)_{0\leq t\leq 1}$ converges weakly as $\sigma \uparrow \infty$ to a Brownian motion B in $\mathrm{LIE}(\mathscr{D})$ with **covariance***

$$\Theta(f) := 2\int_0^\infty \mathbb{E}_\mu\Big[f(u_0)\, f(u_t)\Big]\,dt, \qquad f \in \mathrm{LIE}(\mathscr{D})'$$

5. Kinetic Brownian motion in the diffeomorphism group

Set $U_t^\sigma := u_{\sigma^2 t} \in \mathrm{LIE}(\mathscr{D})$. Wlog $\Sigma = \mathrm{diag}(\alpha_i^2)$, non-increasing eigenvalues α_i.

► Theorem (Angst, Bailleul, Perruchaud 19') – **Homogenization in LIE($\mathscr{D}$).** Assume $3\,\alpha_1^2 < \mathrm{tr}(\Sigma)$ – there is sufficient noise in the system.

- *The velocity process in the unit sphere of* LIE($\mathscr{D}$) *is ergodic and converges exponentially fast to its unique invariant measure in* 2*-Wasserstein distance.*
- *The invariant measure* μ *of the velocity process* $\dot{u}$ *on the unit sphere of* LIE($\mathscr{D}$) *is the image by the radial projection on the sphere of the measure on* LIE($\mathscr{D}$) *with density* $|u|^{-1}$ *wrt the Gaussian measure with covariance* Σ*.*
- *The time-rescaled process* $(U_t^\sigma)_{0\leq t\leq 1}$ *converges weakly as* $\sigma \uparrow \infty$ *to a Brownian motion* B *in* LIE($\mathscr{D}$) *with* ***covariance***

$$\Theta(f) := 2\int_0^\infty \mathbb{E}_\mu\Big[f(u_0)\,f(u_t)\Big]\,dt, \qquad f \in \mathrm{LIE}(\mathscr{D})'$$

- The rough path lift $\mathbf{U}^\sigma$ of $(U_t^\sigma)_{0\leq t\leq 1}$ converges to the Stratonovich Brownian rough path associated with B.

5. Kinetic Brownian motion in the diffeomorphism group

Set $U^\sigma_t := u_{\sigma^2 t} \in \mathrm{Lie}(\mathscr{D})$. Wlog $\Sigma = \mathrm{diag}(\alpha_i^2)$, non-increasing eigenvalues α_i.

▶ Theorem (Angst, Bailleul, Perruchaud 19') – **Homogenization in $\mathrm{Lie}(\mathscr{D})$.** Assume $3\,\alpha_1^2 < \mathrm{tr}(\Sigma)$ – there is sufficient noise in the system.

- *The velocity process in the unit sphere of* $\mathrm{Lie}(\mathscr{D})$ *is ergodic and converges exponentially fast to its unique invariant measure in* 2*-Wasserstein distance.*
- *The invariant measure* μ *of the velocity process* $\dot{u}$ *on the unit sphere of* $\mathrm{Lie}(\mathscr{D})$ *is the image by the radial projection on the sphere of the measure on* $\mathrm{Lie}(\mathscr{D})$ *with density* $|u|^{-1}$ *wrt the Gaussian measure with covariance* Σ*.*
- *The time-rescaled process* $(U^\sigma_t)_{0\le t\le 1}$ *converges weakly as* $\sigma \uparrow \infty$ *to a Brownian motion* B *in* $\mathrm{Lie}(\mathscr{D})$ *with* ***covariance***

$$\Theta(f) := 2\int_0^\infty \mathbb{E}_\mu\Big[f(u_0)\,f(u_t)\Big]\,dt, \qquad f \in \mathrm{Lie}(\mathscr{D})'$$

- The rough path lift $\mathbf{U}^\sigma$ of $(U^\sigma_t)_{0\le t\le 1}$ converges to the Stratonovich Brownian rough path associated with B.

About the proof – Convergence results rely on quantifying the speed of decorrelation of the velocity process $\dot{u}$. Not easy in infinite dimension. Use of conditioning and decorrelation speed to get uniform estimates

$$\sup_{\sigma>0} \mathbb{E}\Big[\big\|X^\sigma_t - X^\sigma_s\big\|^p \vee \big\|\mathbb{X}^\sigma_{ts}\big\|^{p/2}\Big] \lesssim_p |t-s|^{p/2}.$$

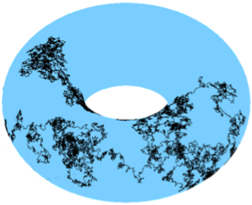
5. Kinetic Brownian motion in the diffeomorphism group

Using the above mentioned version of Cartan's development machinery, one can define kinetic Brownian motion in $\mathscr{D}$ in a *small time interval* by solving a rough differential equation driven by the $\textsc{Lie}(\mathscr{D})$-valued kinetic Brownian motion.
(Recall $\mathscr{D}$ may be geodesically incomplete and may have finite diameter!)

5. Kinetic Brownian motion in the diffeomorphism group

Using the above mentioned version of Cartan's development machinery, one can define kinetic Brownian motion in $\mathscr{D}$ in a *small time interval* by solving a rough differential equation driven by the $\text{LIE}(\mathscr{D})$-valued kinetic Brownian motion. (Recall $\mathscr{D}$ may be geodesically incomplete and may have finite diameter!)

- Theorem (Angst, Bailleul, Perruchaud 19') – **Homogenization in $\mathscr{D}$.** *Kinetic Brownian motion in $\mathscr{D}$ provides an **interpolation** between the dynamics of a(n incompressible) fluid ($\sigma = 0$) and the projection on the diffeomorphism group of a Brownian flow on a larger space ($\sigma = \infty$).*



Thank you!