Path development and the Length Conjecture

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Motivation

Let $\gamma : [0, L] \to \mathbb{R}^d$ be a continuous path with **bounded variation**, parametrised at **unit speed**.

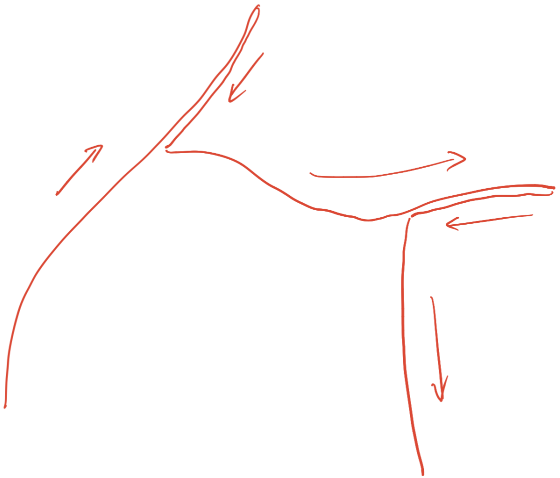
The *signature* of γ is defined as:

$$S(\gamma) = \sum_{n=0}^{\infty} \int_{0<t_1<\cdots<t_n<L} d\gamma_{t_1} \otimes \cdots \otimes d\gamma_{t_n}$$

Theorem (Hambly-Lyons 2010)

The path γ is uniquely determined by its signature $S(\gamma)$ up to tree-like piece.

Tree-like Pieces



Tree-reduced Paths

A *tree-reduced path* is a path that does not contain any tree-like pieces.

- Among all BV paths with the same signature g, there is a **unique tree-reduced representative** (up to reparametrisation).
- This tree-reduced path has the minimal length among all paths with the same signature.
- There is a one-to-one correspondence between tree-reduced BV paths and their signatures.

> In principle, parametrisation-free properties of a tree-reduced path should be **reconstructable** from the knowledge of its signature.

The Main Question

By using the triangle inequality,

$$\begin{aligned}
&\left\| \int_{0<t_1<\cdots<t_n<L} d\gamma_{t_1} \otimes \cdots \otimes d\gamma_{t_n} \right\| \\
&\leqslant \int_{0<t_1<\cdots<t_n<L} |\dot{\gamma}_{t_1}| \cdots |\dot{\gamma}_{t_n}| dt_1 \cdots dt_n \\
&= \int_{0<t_1<\cdots<t_n<L} dt_1 \cdots dt_n \\
&= \frac{L^n}{n!}.
\end{aligned}$$

The Main Question

By rearrangement,

$$\left(n!\|\pi_n(S(\gamma))\|\right)^{1/n} \leqslant L \quad \forall n \geqslant 1.$$

If the path is tree-reduced, the above inequality (surprisingly!) becomes asymptotically sharp as $n \to \infty$.

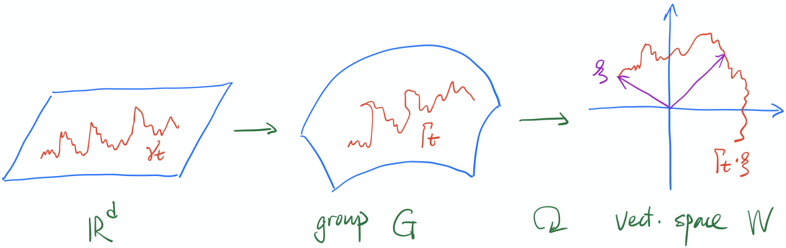
Conjecture (Hambly-Lyons 2010, Chang-Lyons-Ni 2018)

For any tree-reduced BV path γ,

$$\mathit{Length}(\gamma) = \lim_{n\to\infty} \left(n!\|\pi_n(S(\gamma))\|\right)^{1/n}.$$

Path Developments

General philosophy: By lifting a path/stochastic process onto a suitably chosen Lie group and looking at how the lifted object acts on a well chosen space (representation theory), one can detect quantitative properties of the original path/process.



Path Developments

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let W be a finite dimensional vector space.

- G *acts on* W if there is a continuous map $\pi : G \longrightarrow \mathrm{Aut}(W)$ such that
$$\pi(gh)w = \pi(g)\pi(h)w \quad \forall g, h \in G, \ w \in W.$$
- π is called a *representation* of G over W.
- A *representation* of the Lie algebra $\mathfrak{g}$ over W is a Lie algebra homomorphism $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(W)$:
$$\rho([X, Y])w = [\rho(X), \rho(Y)]w \quad \forall X, Y \in \mathfrak{g}, \ w \in W.$$

Path Developments

- A group representation $\pi : G \longrightarrow \mathrm{Aut}(W)$ induces a Lie algebra representation $d\pi : \mathfrak{g} \longrightarrow \mathrm{End}(W)$:

$$X \cdot w \triangleq \left.\frac{d}{dt}\right|_{t=0} (\exp tX) \cdot w, \quad X \in \mathfrak{g}, w \in W.$$

- Conversely, if G is simply connected, then any representation of $\mathfrak{g}$ induces a representation of G (**Lie's theorem**).

Representation theory:

- Study Lie groups and Lie algebras by representing their elements as linear transformations over vector spaces.

Path Developments

Let $\mathbf{X}_t$ be a geometric rough path over $\mathbb{R}^d$. Let G be a Lie group with Lie algebra $\mathfrak{g}$.

- Fix a linear map $F : \mathbb{R}^d \longrightarrow \mathfrak{g}$.
- The *Cartan development* of $\mathbf{X}_t$ onto G under F is the solution to the following differential equation:

$$\begin{cases} d\Gamma_t = \Gamma_t \cdot F(d\mathbf{X}_t), \\ \Gamma_0 = e. \end{cases}$$

Path Developments

The signature path of $\mathbf{X}_t$ is a particular type of Cartan's development:

- $G = G^N(\mathbb{R}^d)$, $\mathfrak{g} = \mathfrak{g}^N(\mathbb{R}^d)$.
- $F : \mathbb{R}^d \to \mathfrak{g}^N(\mathbb{R}^d)$ is the natural embedding.
- The signature differential equation:

$$dS_N(\mathbf{X})_{0,t} = S_N(\mathbf{X})_{0,t} \otimes d\mathbf{X}_t.$$

General Cartan's development:

$$\Gamma_t = \mathrm{Id} + \sum_{n=1}^{\infty} F^{\otimes n}\Big(\int_{0<t_1<\cdots<t_n<t} d\mathbf{X}_{t_1} \otimes \cdots \otimes d\mathbf{X}_{t_n}\Big).$$

Let $\pi : G \to \mathrm{Aut}(W)$ be a representation with a distinguished vector $\xi \in W$.

- The orbit $Y_t \triangleq \Gamma_t \cdot \xi$ encodes rich information about $\mathbf{X}_t$.

Path Developments

Why do we consider path developments?

- When ranging over a suitable class of Lie groups and their representations, the end point of the Cartan development encodes essentially all information about $\mathbf{X}_t$.
- The Cartan development is defined by a linear ODE.
- Representations of classical groups/Lie algebras are well studied and classified.

The Length Conjecture

Returning to the main signature problem:

- $\gamma : [0, L] \to \mathbb{R}^d$ is a tree-reduced BV path with unit speed parametrisation.
- Equip $\mathbb{R}^d$ with the Euclidean norm and $(\mathbb{R}^d)^{\otimes n}$ with the *projective tensor norms*.
- **The ultimate goal**:

$$\mathrm{Length}(\gamma) = \lim_{n\to\infty} \left(n! \left\| \int_{0<t_1<\cdots<t_n<L} d\gamma_{t_1} \otimes \cdots \otimes d\gamma_{t_n} \right\| \right)^{1/n}$$

Piecewise Linear / C^1 Paths: Hyperbolic Development

Theorem (Hambly-Lyons 2010, Lyons-Xu 2015)

Let γ be a tree-reduced BV path with unit speed parametrisation. If γ is either piecewise linear or C^1, then

$$\text{Length}(\gamma) = \lim_{n\to\infty}\left(n!\left\|\int_{0<t_1<\cdots<t_n<L} d\gamma_{t_1}\otimes\cdots\otimes d\gamma_{t_n}\right\|\right)^{1/n}$$

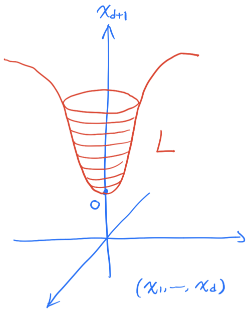
Piecewise Linear / C^1 Paths: Hyperbolic Development

Strategy: Hyperbolic development [Hambly-Lyons 2010].

- G: a group of isometries for the hyperbolic space with constant negative curvature -1.

The hyperboloid model:

$$L \triangleq \{x \in \mathbb{R}^{d+1} : x_1^2 + \cdots + x_d^2 - x_{d+1}^2 = -1, x_{d+1} > 0\}$$



Piecewise Linear / C^1 Paths: Hyperbolic Development

The **isometry group** of L:

$$G = \left\{ M \in \mathrm{Mat}(d+1;\mathbb{R}) : MJM^T = J \right\} \text{ where } J \triangleq \begin{pmatrix} \mathrm{I}_d & 0 \\ 0 & -1 \end{pmatrix}$$

The **group action**: acting on the hyperboloid by isometry.

The **linear map** $F : \mathbb{R}^d \to \mathfrak{g}$:

$$x \mapsto F(x) \triangleq \begin{pmatrix} 0 & x \\ x^T & 0 \end{pmatrix}, \quad x \in \mathbb{R}^d.$$

The **hyperbolic development**:

$$\gamma_t \in \mathbb{R}^d \mapsto \Gamma_t \in G \mapsto X_t \triangleq \Gamma_t \cdot o \in L.$$

Piecewise Linear / C^1 Paths: Hyperbolic Development

Define the quantity

$$\tilde{L} \triangleq \lim_{n\to\infty} \left(n!\|\pi_n(S(\gamma))\|\right)^{1/n}$$
$$= \sup_{n\geqslant 1} \left(n!\|\pi_n(S(\gamma))\|\right)^{1/n}. \quad \text{(Chang-Lyons-Ni, Boedihardjo-G. 2018)}$$

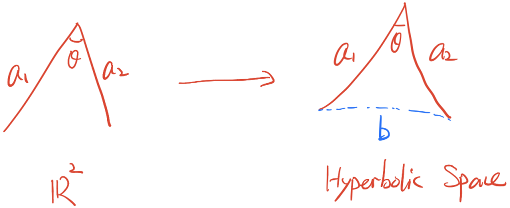
For any $\lambda > 0$, let X_t^λ be the **hyperbolic development** of $\lambda \cdot \gamma_t$.

$$\cosh d_{\mathrm{hyp}}(X_L^\lambda, o) = \sum_{n=0}^{\infty} \lambda^{2n} \int_{0<t_1<\cdots<t_{2n}<L} \langle d\gamma_{t_1}, d\gamma_{t_2}\rangle \cdots \langle d\gamma_{t_{2n-1}}, d\gamma_{t_{2n}}\rangle$$
$$\leqslant \cosh \lambda \tilde{L}.$$

It is sufficient to show:

$$\lambda L - d_{\mathrm{hyp}}(X_L^\lambda, o) \leqslant o(\lambda) \quad \text{as } \lambda \to \infty.$$

The Piecewise Linear Case: Two Edges



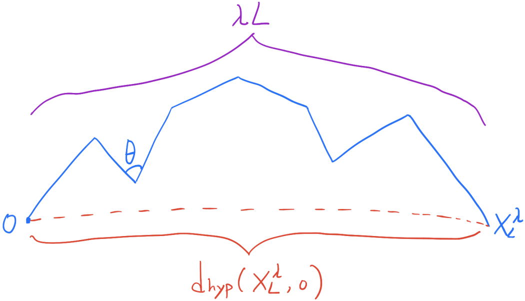
Lemma (Reversed Triangle Inequality)

The following estimate holds:

$$a_1 + a_2 - b \leqslant \log \frac{2}{1 - \cos\theta}.$$

The Piecewise Linear Case: N Edges

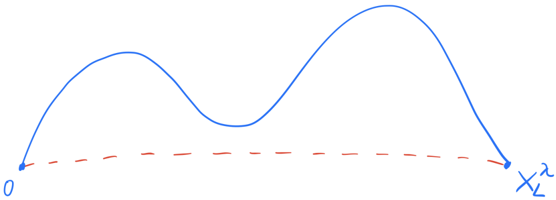
Let $\gamma = v_1 \sqcup \cdots \sqcup v_N$ be a piecewise linear path with N edges and **minimal intersection angle** θ.



$$\lambda L - d_{\mathrm{hyp}}(X_L^\lambda, o) \leqslant N \cdot \log \frac{2}{1 - \cos\theta} \quad \forall \lambda > 0.$$

The C^1 case

Let $\gamma \in C^1$ with unit speed parametrisation.



The following estimate holds:

$$0 \leqslant \lambda L - d_{\text{hyp}}(X_L^\lambda, o) \leqslant C \cdot \lambda \cdot \Phi(\delta_{\dot{\gamma}}(1/\lambda)) \quad \forall \lambda > 0.$$

where $\delta_{\dot{\gamma}}(\cdot)$ is the **modulus of continuity** of $\dot{\gamma}$ and $\Phi(0) = 0$.

Towards the General BV Case

Let $\gamma : [0, L] \to \mathbb{R}^2$ be a BV path parametrised **at unit speed**.

- Express γ_t as

$$\gamma_t = (x_t, y_t) : \begin{cases} x_t = x_0 + \int_0^t \cos \beta_s ds, \\ y_t = y_0 + \int_0^t \sin \beta_s ds. \end{cases}$$

- The angular path β_t is a measurable function.

Towards the General BV Case

A tree-reduced type assumption:

- We say that γ is *strongly tree-reduced* if at each point t there is a neighbourhood U_t such that β_s takes values within some interval of length $< \pi$ a.e. $s \in U_t$.

Theorem (Boedihardjo-G. 2020)

Let $\gamma : [0, L] \to \mathbb{R}^2$ be strongly tree-reduced. Then

$$\mathit{Length}(\gamma) = \lim_{n\to\infty} \left(n! \left\| \int_{0<t_1<\cdots<t_n<L} d\gamma_{t_1} \otimes \cdots \otimes d\gamma_{t_n} \right\| \right)^{1/n}.$$

Towards the General BV Case

Main strategy:

- Develop γ onto the special linear group $\mathrm{SL}_2(\mathbb{R})$.
- Under the canonical linear action, the quantity

$$\tilde{L} = \lim_{n\to\infty} \left(n!\|\pi_n(S(\gamma))\|\right)^{1/n}$$

  is controlled from below by the associated **angle dynamics**.
- Analyse the angle dynamics at a microscopic level.

Main difficulty: The parameters of the equations are only measurable functions.

Main benefit of $\mathrm{SL}_2(\mathbb{R})$: Under the canonical linear action, one obtains a decoupled ODE system in polar coordinates.

Limitation: The argument relies on monotonicity properties crucially.

$\mathrm{SL}_2(\mathbb{R})$ and its Lie Algebra

The **Lie group**:

$$\mathrm{SL}_2(\mathbb{R}) \triangleq \{A \in \mathrm{Mat}(2;\mathbb{R}) : \det(A) = 1\}\,.$$

The **Lie algebra** of $\mathrm{SL}_2(\mathbb{R})$:

$$\mathfrak{sl}_2(\mathbb{R}) \triangleq \{A \in \mathrm{Mat}(2;\mathbb{R}) : \mathrm{Tr}(A) = 0\}\,.$$

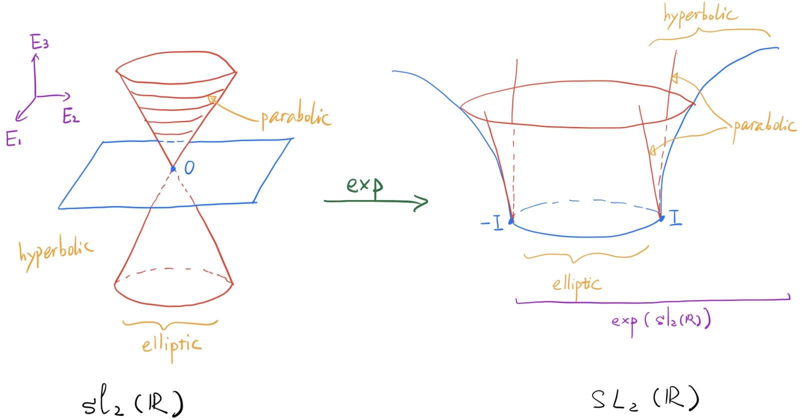
- A natural Lorentzian metric on $\mathfrak{sl}_2(\mathbb{R})$:

$$\langle A, B\rangle \triangleq \frac{1}{2}\mathrm{Tr}(AB), \quad A, B \in \mathfrak{sl}_2(\mathbb{R}).$$

- An element $A \in \mathfrak{sl}_2(\mathbb{R})$ is hyperbolic/elliptic/parabolic if $\langle A, A\rangle$ is positive/negative/zero.
- A canonical ONB of $\mathfrak{sl}_2(\mathbb{R})$:

$$E_1 \triangleq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ E_2 \triangleq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ E_3 \triangleq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$SL_2(\mathbb{R})$ and its Lie Algebra



The $SL_2(\mathbb{R})$-development

We choose:

- $F : \mathbb{R}^2 \longrightarrow \mathfrak{sl}_2(\mathbb{R})$, $F(e_i) \triangleq E_i$ $(i = 1, 2)$:

$$(x, y) \mapsto \begin{pmatrix} x & y \\ y & -x \end{pmatrix}.$$

- The group action is the standard linear transformation over $\mathbb{R}^2$.

For each $\lambda > 0$, define Γ_t^λ to be the Cartan development of $\lambda \cdot \gamma$ under F.

An intermediate lower estimate:

$$\tilde{L} \triangleq \lim_{n\to\infty} \left(n!\|\pi_n(S(\gamma))\|\right)^{1/n} \geqslant \sup_{\xi\in S^1} \overline{\lim_{\lambda\to\infty}} \frac{\log \left|\Gamma_L^\lambda \cdot \xi\right|_{\mathbb{R}^2}}{\lambda}.$$

The $SL_2(\mathbb{R})$-development

Fix $\xi \in S^1$.

We can think of $\Gamma_L^\lambda \cdot \xi$ as the endpoint of the dynamics:

$$w_t^\lambda \triangleq \Gamma_{L-t,L}^\lambda \cdot \xi, \quad t \in [0, L]$$

where $\Gamma_{u,v}^\lambda$ denotes the Cartan development of $\lambda \cdot \gamma|_{[u,v]}$.

- The polar coordinates of $w_t^\lambda = \rho_t^\lambda e^{i\phi_t^\lambda}$ satisfies:

$$\begin{cases} \dot{\rho}_t^\lambda = \lambda \rho_t^\lambda \cos(\alpha_t - 2\phi_t^\lambda), \\ \dot{\phi}_t^\lambda = \lambda \sin(\alpha_t - 2\phi_t^\lambda). \end{cases} \qquad (\alpha_t \triangleq \beta_{L-t})$$

Analysis of the Angle Dynamics

The **angle dynamics**:

$$\dot{\phi}_t^\lambda = \lambda \sin(\alpha_t - 2\phi_t^\lambda).$$

The **radial dynamics**:

$$\rho_t^\lambda = \exp\left(\lambda \int_0^L \cos(\alpha_t - 2\phi_t^\lambda) dt\right).$$

The previous intermediate lower estimate implies:

$$\tilde{L} \geqslant \overline{\lim_{\lambda \to \infty}} \int_0^L \cos(\alpha_t - 2\phi_t^\lambda) dt.$$

The core ingredient:

- When λ is large, $2\phi_t^\lambda$ and α_t are sufficiently close for most of the time.
- **Main challenge**: α_t is only a measurable function.

The Angle Dynamics: $\dot{\phi}_t^\lambda = \lambda \sin(\alpha_t - 2\phi_t^\lambda)$

Lemma

Let $I \triangleq [a, a + \pi]$. Suppose that $\alpha_t \in I$ for all $t \in [0, L]$. Then

$$2\phi_0^\lambda \in I \implies 2\phi_t^\lambda \in I \quad \forall t \in [0, L].$$

Heuristics: If

- α_t good + $2\phi_0^\lambda$ good $\implies$ $2\phi_t^\lambda$ good.

The Angle Dynamics: $\dot{\phi}_t^\lambda = \lambda \sin(\alpha_t - 2\phi_t^\lambda)$

Lemma

Let $[a, b]$ be an interval of length $< \pi$. Then

$$2\phi_0^\lambda \in [a, b] \implies 2\phi_t^\lambda \in [a - r, b + r] \quad \forall t \in [0, L],$$

where $r \triangleq 2\lambda\mu\{t : \alpha_t \notin [a, b]\}$ and μ denotes the Lebesgue measure.

Heuristics: If

- α_t remains in $[a, b]$ for most of the time
- and $2\phi_0^\lambda \in [a, b]$,

then $2\phi_t^\lambda$ does not deviate too much from $[a, b]$.

The Angle Dynamics: $\dot{\phi}_t^\lambda = \lambda \sin(\alpha_t - 2\phi_t^\lambda)$

Let $[c-\varepsilon, d+\varepsilon] \subseteq (a,b)$ where $b-a<\pi$. Define

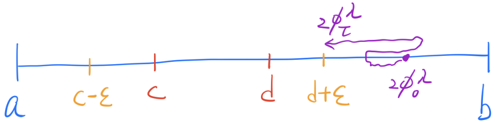
$$\tau \triangleq \inf\{t : 2\phi_t^\lambda \in [c-\varepsilon, d+\varepsilon]\}.$$

Lemma

Suppose that $2\phi_0^\lambda \in (a,b)\backslash[c-\varepsilon, d+\varepsilon]$. *Then*

$$\tau \leqslant \frac{b-a}{2\lambda \sin\varepsilon} + \frac{1+\sin\varepsilon}{\sin\varepsilon}\mu(B^c),$$

where $B \triangleq \{t : \alpha_t \in [c,d]\}$.



Main Steps for Proving the Main Theorem

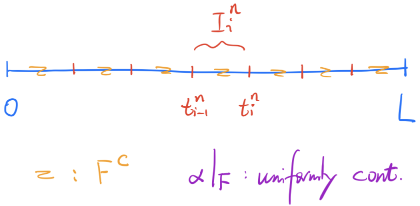
By Lusin's theorem, there exists a compact subset $F \subseteq [0, L]$, such that

- $\mu(F^c) < \eta$;
- there exists $\rho > 0$ such that

$$s, t \in F, \ |t - s| < \rho \implies |\alpha_t - \alpha_s| < \varepsilon.$$

Partition $[0, L]$ into sub-intervals

$$[0, L] = \cup_{i=1}^{n} I_i^n, \quad |I_i^n| < \rho.$$

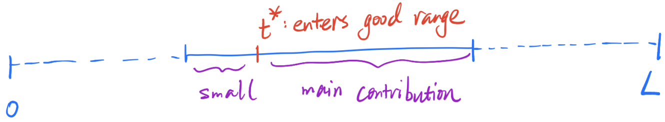


Main Steps for Proving the Main Theorem

Over each sub-interval I_i^n, we define:

$$\alpha_i^n \triangleq \inf\{\alpha_t : t \in F \cap I_i^n\},\ \beta_i^n \triangleq \sup\{\alpha_t : t \in F \cap I_i^n\}.$$

- $[\alpha_i^n, \beta_i^n]$ is the effective range of α on I_i^n.



Main steps of the proof:

1. The time it takes $2\phi_t^\lambda$ $(t \in I_i^n)$ to enter the "good region" $[\alpha_i^n - \varepsilon, \beta_i^n + \varepsilon]$ adds up (over i) to a negligible quantity.
2. Once $2\phi_t^\lambda \in [\alpha_i^n - \varepsilon, \beta_i^n + \varepsilon]$ at some $t^* \in I_i^n$, the portion $[t^*, t_i^n]$ provides a main contribution in the key lower estimate of $\tilde{L}$.

Main Steps for Proving the Main Theorem

Recall:

$$\tilde{L} \geqslant \overline{\lim_{\lambda\to\infty}} \int_0^L \cos(\alpha_t - 2\phi_t^\lambda)dt.$$

We shall write

$$\int_0^L \cos(\alpha_t - 2\phi_t^\lambda)dt$$

$$= \int_{\{\text{times before entering good regions}\}} \qquad \text{(Negligible)}$$

$$+ \int_{\{\text{times when staying in good regions}\}} \qquad \text{(Main Contribution)}$$

Main Steps for Proving the Main Theorem

$$\int_0^L \cos(\alpha_t - 2\phi_t^\lambda)dt$$

$$\geqslant \cos\left(2\varepsilon + \frac{2M\eta}{\varepsilon \sin\varepsilon}\right)(L - \delta - \eta) - \delta - \eta - \frac{\pi\varepsilon}{2} - \frac{L}{M} - \frac{(1+\sin\varepsilon)\eta}{\sin\varepsilon}$$

$$-\frac{L}{M}\cos\left(2\varepsilon + \frac{2M\eta}{\varepsilon \sin\varepsilon}\right) - \cos\left(2\varepsilon + \frac{2M\eta}{\varepsilon \sin\varepsilon}\right)\cdot\left(\frac{\pi\varepsilon}{2} + \frac{(1+\sin\varepsilon)\eta}{\sin\varepsilon}\right).$$

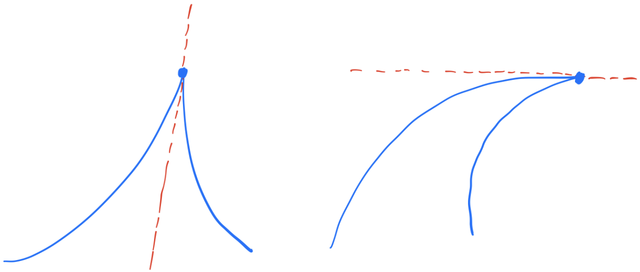
and pass to the limit strictly in the following order:

$$\lambda \to \infty, \;\; \eta \to 0, \;\; M \to \infty, \;\; \varepsilon \to 0, \;\; \delta \to 0!$$

$$\Longrightarrow \;\; \tilde{L} \geqslant \overline{\lim_{\lambda\to\infty}} \int_0^L \cos(\alpha_t - 2\phi_t^\lambda)dt \geqslant L.$$

Extensions

The argument extends to deal with paths with **cusp singularities**:



The Rough Path Situation

Let $\mathbf{X}_t$ be a **geometric rough path** with roughness p.

- Lyons' extension theorem:

$$\left\|\pi_n(S(\mathbf{X}))\right\| \leqslant \frac{\omega(\mathbf{X})^{n/p}}{(n/p)!} \quad \forall n \geqslant 1.$$

Question: What does the quantity

$$\overline{\lim_{n\to\infty}} \left((n/p)! \left\|\pi_n(S(\mathbf{X}))\right\|\right)^{p/n}$$

recover?

Conjectural answer: some sort of local p-variation of $\mathbf{X}$.

The Rough Path Situation

Theorem (Boedihardjo-G.-Souris 2020)

For each m, there exists a constant $C_m \in (0,1)$ such that

$$C_m \|\mathbf{X}\|_{local\text{-}m\text{-}var} \leqslant \overline{\lim_{n\to\infty}} \left((n/m)! \left\|\pi_n(S(\mathbf{X}))\right\|\right)^{m/n} \leqslant \|\mathbf{X}\|_{local\text{-}m\text{-}var}.$$

for all $\mathbf{X}_t = \exp(tl)$ where l is an arbitrary Lie polynomial of degree m. If $m = 2, 3$, we have $C_m = 1$.

Main idea of the proof:

- Develop the path onto semisimple Lie groups.
- The lower bound is reduced to the study of spectral properties of l under the action $\rho : \mathfrak{g} \to \mathrm{End}(W)$.
- This can be studied effectively by using the representation theory of semisimple Lie algebras.

Open Questions

1. Can we extend the analysis to **higher dimensions**?
2. How far is it towards a **complete proof** (or counterexample) of the length conjecture?
3. How about the **rough path situation**?
4. Can we apply the idea of path developments to the study of **other signature inversion properties**?

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