Signatures and Functional Expansions

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DataSig Seminar Series

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Joint work with Valentin Tissot-Daguette (Bloomberg / Princeton)

OUTLINE

- Signatures
- Functional Itô Calculus
- Expansions of Functionals
  - Wiener Chaos
  - Intrinsic Value
  - Functional Taylor
- Applications
  - Claim Decomposition
  - Hedging with Signature

(t, x) SIGNATURES

ONE DIMENSION + TIME

For the sake of simplicity we consider the one asset case and its price time series. It is described by (t, x_t)

- The signatures are iterated (Stratonovich) integrals with respect to the variables t and x.
- They are described by a word with letters in the alphabet $\{t, x\}$
- We denote them as binary strings with the convention $t \mapsto 0$ and $x \mapsto 1$

For instance the word “$txttx$” becomes “01001”, which corresponds to the following integral:

$$S_{01001}(X_T) = \int_0^T \int_0^{t_5} \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} \circ\, dt_1 \circ dx_{t_2} \circ dt_3 \circ dt_4 \circ dx_{t_5}$$

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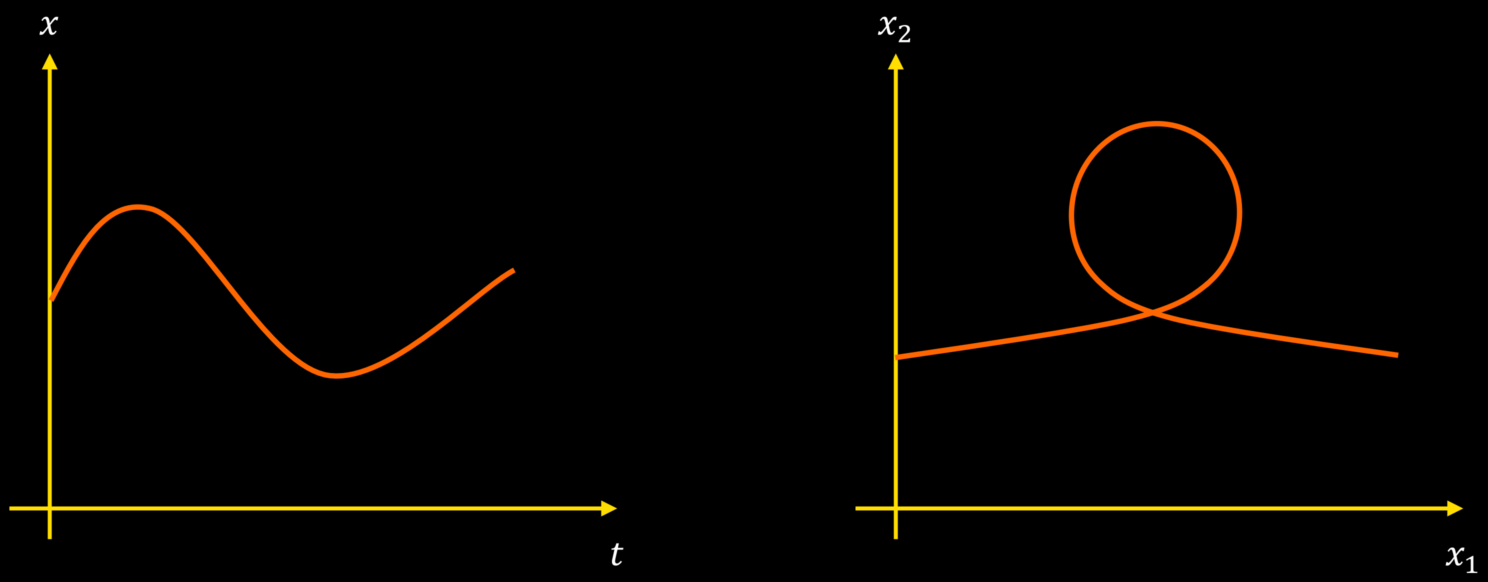
(T,X) PATH VS (X1,X2) PATH



$x_t = x(t)$

$x_{1,t} = x_1(t)$
$x_{2,t} = x_2(t)$

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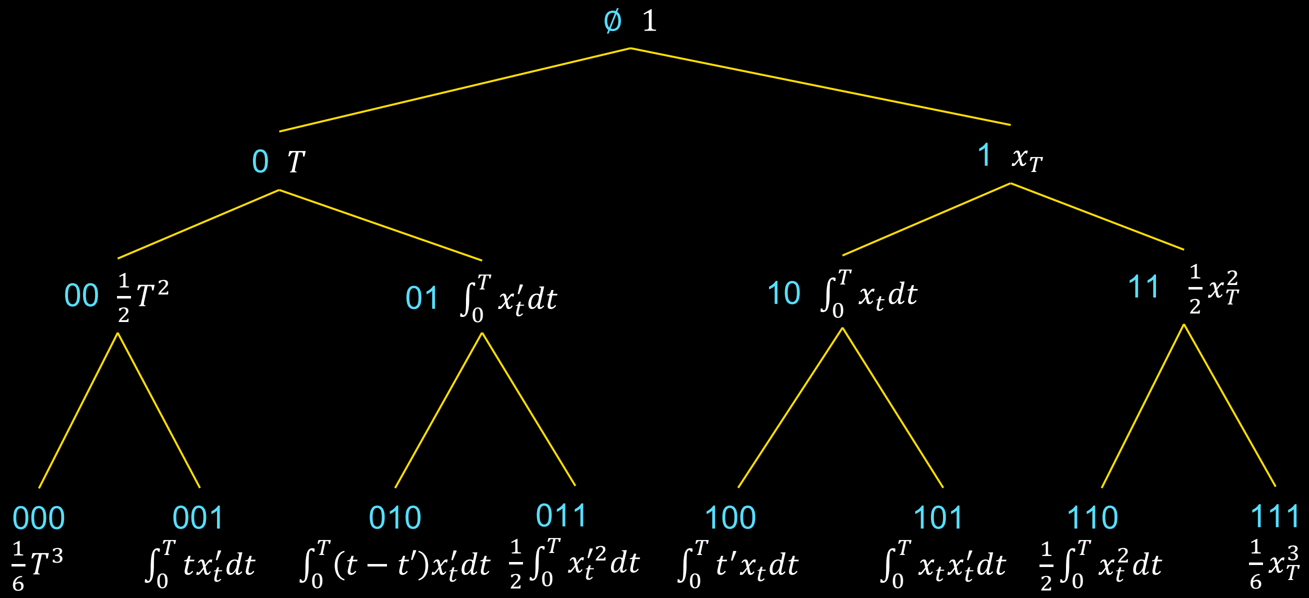
SHORT WORDS EXAMPLES

Assume: $x_0 = 0$

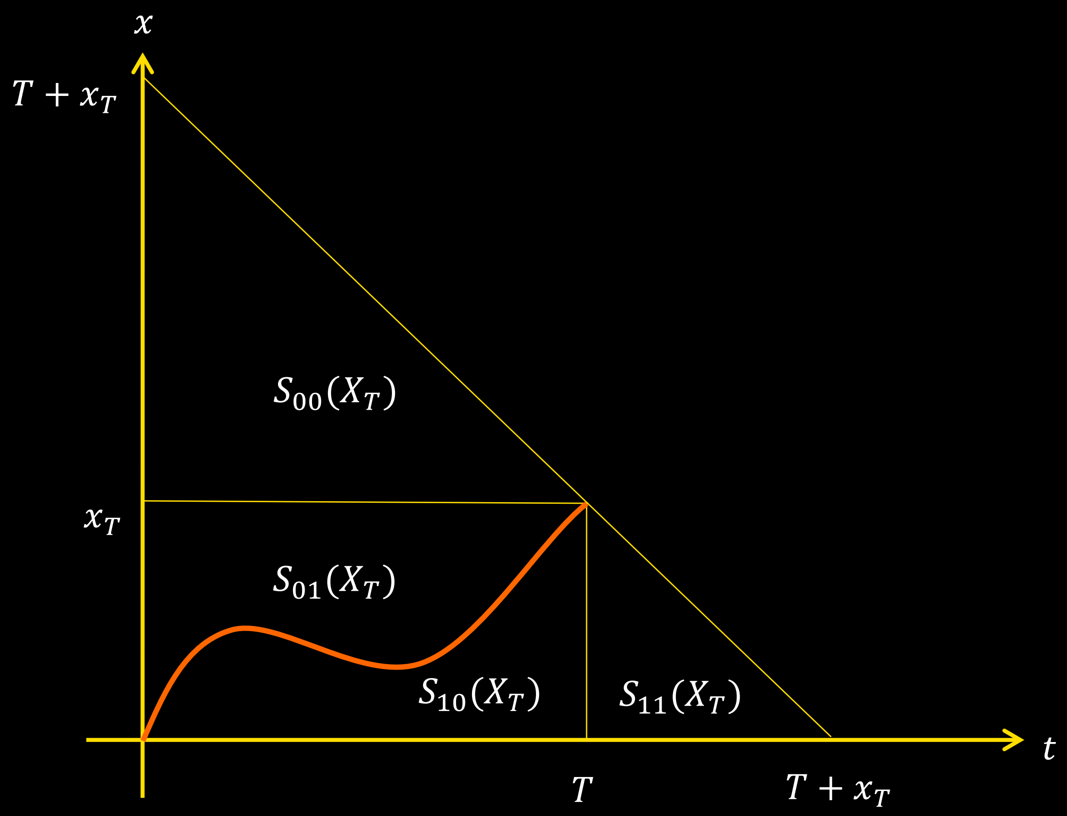
Define:

$$t' = T - t$$

$$x'_t = x_T - x_t$$

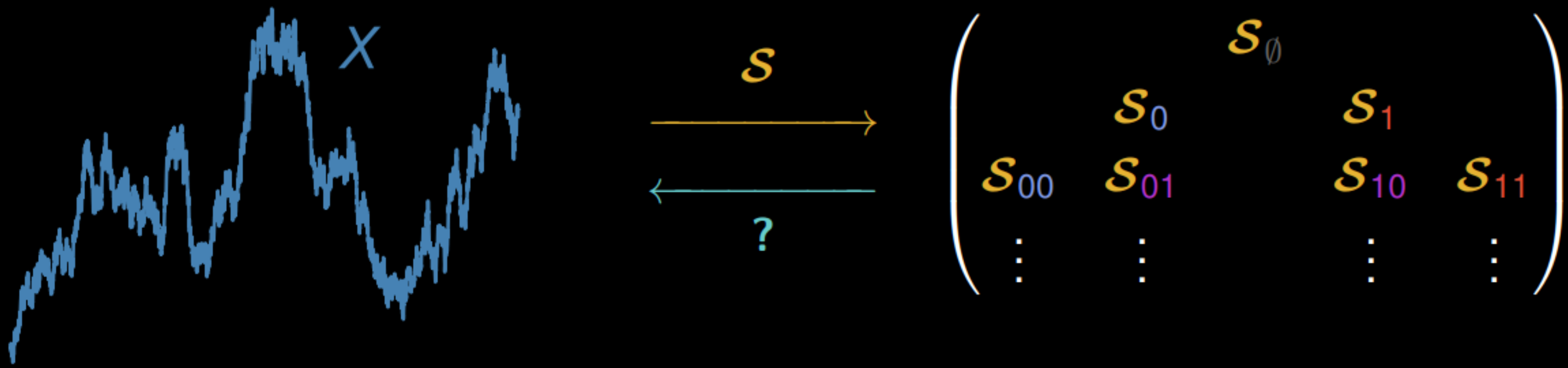


SIGNATURES: GEOMETRIC VISUALISATION



PATH RECONSTRUCTION

PATH RECONSTRUCTION



- We can compute signatures from a path.
- Can we reconstruct the path from the signatures?

Yes, and a subset of words is enough

RECONSTRUCTION PROPERTY

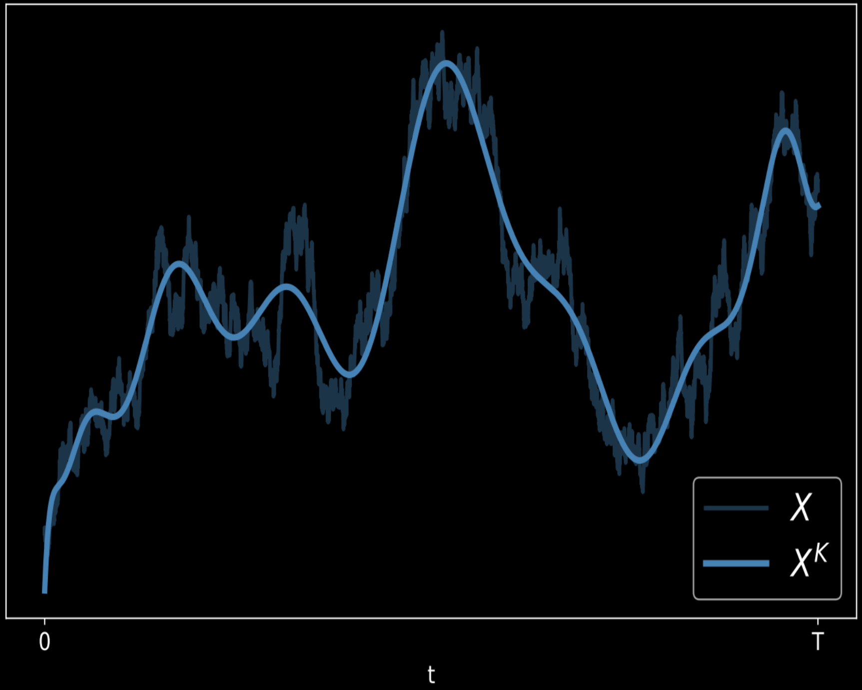
- Legendre words of length $k+1$:

  $\alpha = (1, \underbrace{0,0,\dots,0}_{k})$:

  $$S_\alpha(X_T) = \int_0^T x_s \frac{(T-s)^{k-1}}{(k-1)!} ds$$

  give L^2 product of path with polynomials in t

- They are enough to rebuild the path



Further details in [V. Tissot-Daguette. "Short communication: Projection of functionals and fast pricing of exotic options", *SIFIN*, 2022]

FUNCTIONAL ITÔ CALCULUS

FUNCTIONAL ITÔ CALCULUS

Paper available at: https://papers.ssrn.com/sol3/Papers.cfm?abstract_id=1435551

- Calculus for functions of the path so far, not only functions of the current value
- Proper definition of Greeks for path dependent options
- Functional Itô formula gives Γ/Θ trade-off for path-dependent options

RESULTS AND APPLICATIONS

- Functional versions of Itô formula, Feynman-Kacs and BS PDE

- Super-replication (refinement of Kramkov decomposition)

- Lie Bracket of price and time functional derivatives

- Characterisation of attainable claims

- Decomposition of volatility risk

REVIEW OF ITÔ CALCULUS

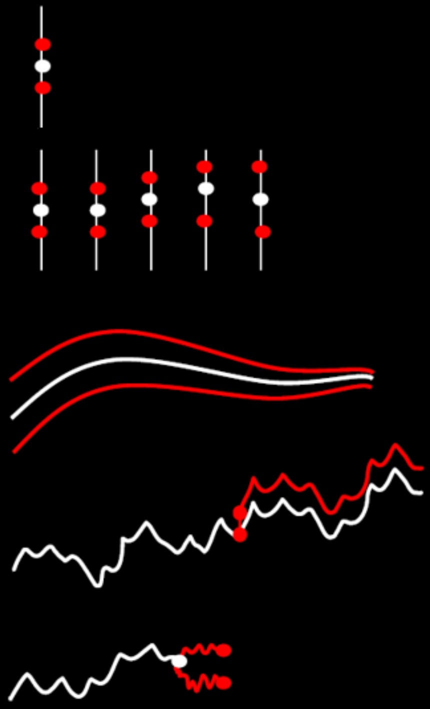
1D

nD

infiniteD

Malliavin Calculus

Functional Itô Calculus



- current value
- possible evolutions

PATH SPACES

$\Lambda_t = \{\text{càdlàg paths over } [0, t]\}$

$\Lambda = \bigcup_{t \in [0,T]} \Lambda_t$

$X_t = \{X_t(s), s \in [0, t]\} \in \Lambda_t$ $\quad x_s = X_t(s) \in \mathbb{R}$

FUNCTIONALS, T-FUNCTIONALS

T-functional:

$$g: \Lambda_T \mapsto \mathbb{R}$$

$$X_T \mapsto g(X_T)$$

A T-functional can be seen as a random variable in the Wiener space, or as a payoff of an exotic (path dependent) option.

Functional:

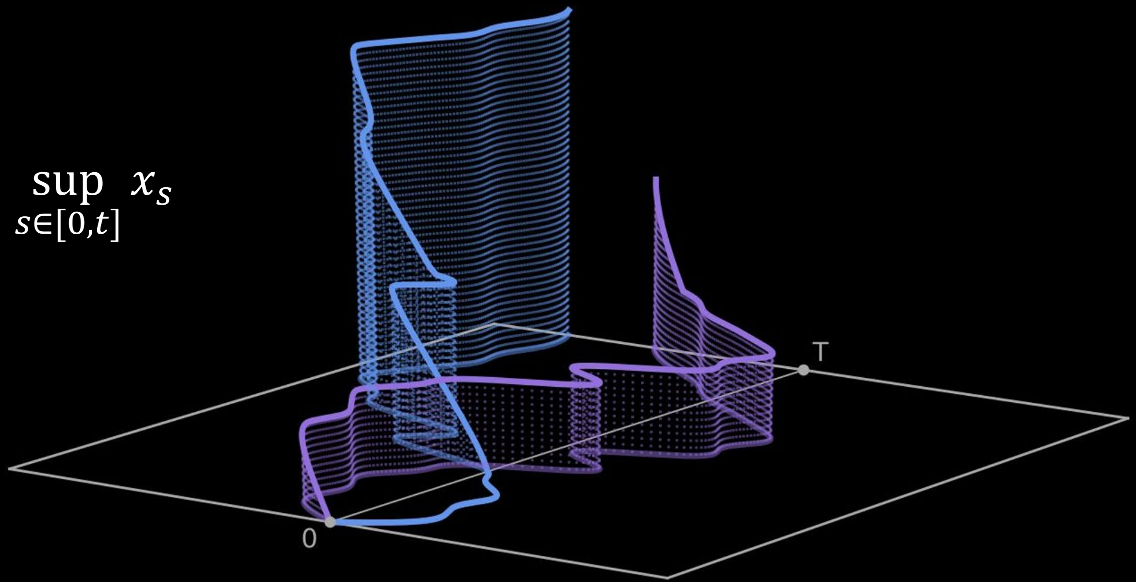
$$f: \Lambda \mapsto \mathbb{R}$$

$$X_t \mapsto f(X_t)$$

An example of a functional is the price of an exotic option, knowing X_t (the underlying price path) so far: $f(X_t) = \mathbb{E}^{\mathbb{Q}}[g(Y_T)|X_t]$

EXAMPLE OF A FUNCTIONAL

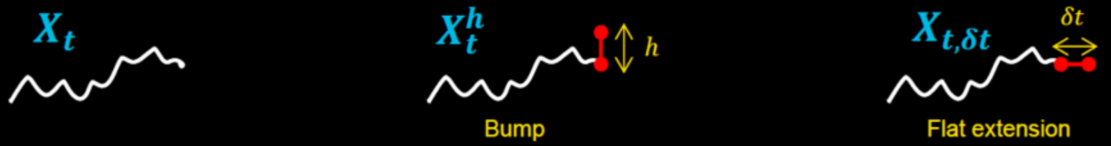
$$f(X_t) = \sup_{s \in [0,t]} x_s$$



f on two paths

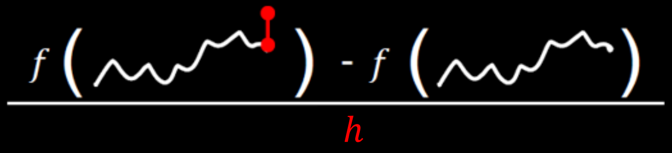
FUNCTIONAL DERIVATIVES

Two Operators:



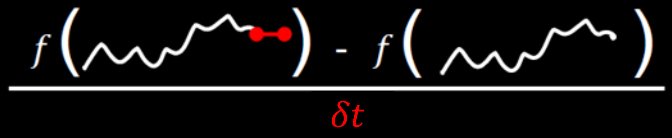
Space derivative

$$\Delta_x f(X_t) \equiv \lim_{h \to 0} \frac{f(X_t^h) - f(X_t)}{h} \equiv \lim_{h \to 0}$$

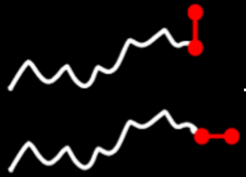


Time derivative

$$\Delta_t f(X_t) \equiv \lim_{\delta t \to 0^+} \frac{f(X_{t,\delta t}) - f(X_t)}{\delta t} \equiv \lim_{\delta t \to 0^+}$$



EXAMPLES



f	x_t	$\int_0^t x_s ds$	$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T x_s ds \mid X_t\right]$	$\langle X \rangle_t$
$\Delta_x f$	1	0	$T - t$	$2(x_t - x_{t-})$
$\Delta_t f$	0	x_t	0	0

- $f(X_t) = h(t, x_t) \implies \Delta_x f = \partial_x h, \ \Delta_t f = \partial_t h$
- $f(X_t) = \int_0^t x_s ds \implies 0 = \Delta_{tx} f \neq \Delta_{xt} f = 1$

PATHWISE ITÔ & STRATONOVICH FORMULA

- **Föllmer's approach**: Given sequence of refining partitions $\Pi = (\Pi^N)_{N \geq 1}$ of $[0, T]$

- If $X \in \mathbf{\Omega}^{\Pi} = \left\{ \text{continuous}, \langle X \rangle_t{}^{\Pi} = \lim_{N \to \infty} \sum_{t_n \in \Pi^N} \left(x_{t_n \wedge t} - x_{t_{n-1} \wedge t} \right)^2 \text{ exists \& finite} \right\}$, then

  - **Functional Itô formula** ($f \in \mathbb{C}^{1,2}$)

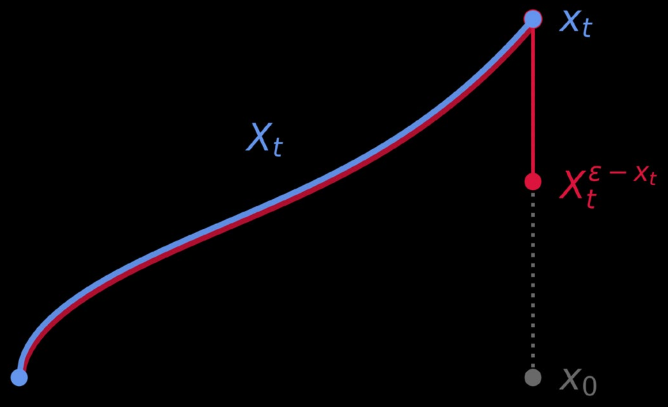
$$f(X_t) = f(X_0) + \int_0^t \Delta_t f(X_s) ds + \int_0^t \Delta_x f(X_s) dx_s + \frac{1}{2} \int_0^t \Delta_{xx} f(X_s) d\langle x \rangle_s$$

  - **Functional Stratonovich formula** ($f \in \mathbb{C}^{1,2}$)

$$f(X_t) = f(X_0) + \int_0^t \Delta_t f(X_s) ds + \int_0^t \Delta_x f(X_s) \circ dx_s$$

PATHWISE STRATONOVICH INTEGRATION

- **Goal:** define $\int_0^t h(X_s) \circ dx_s$ for all $h \in \mathbb{C}^{1,1}$ and $X \in \Omega^{\Pi}$
- **Spatial anti-derivative:** $H(X_t) = \int_{x_0}^{x_t} h\left(X_t^{\epsilon - x_t}\right) d\epsilon$
  - $\Delta_x H = h$
  - $\Delta_t H(X_t) = \int_{x_0}^{x_t} \Delta_t h\left(X_t^{\epsilon - x_t}\right) d\epsilon$



- Functional Stratonovich formula (rearranged):

$$\int_0^t h(X_s) \circ dx_s = H(X_t) - \int_0^t \Delta_t H(X_s)\, ds$$

- If $h = S_\alpha \Longrightarrow$ one can define $S_{\alpha 1}$ in a pathwise manner (in fact, $S = (S_\alpha)$ entirely)

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FUNCTIONAL DERIVATIVES OF INTEGRALS

f	$\int_0^t h(X_s)ds$	$\int_0^t h(X_s)dx_s$	$\int_0^t h(X_s) \circ dx_s$
Integral type	Riemann	Itô	Stratonovich
$\Delta_t f$	$h(X_t)$	0	0
$\Delta_x f$	0	$h(X_{t-})$	$h(X_t)$
$\Delta_{xx} f$	0	0	$\Delta_x h(X_t)$

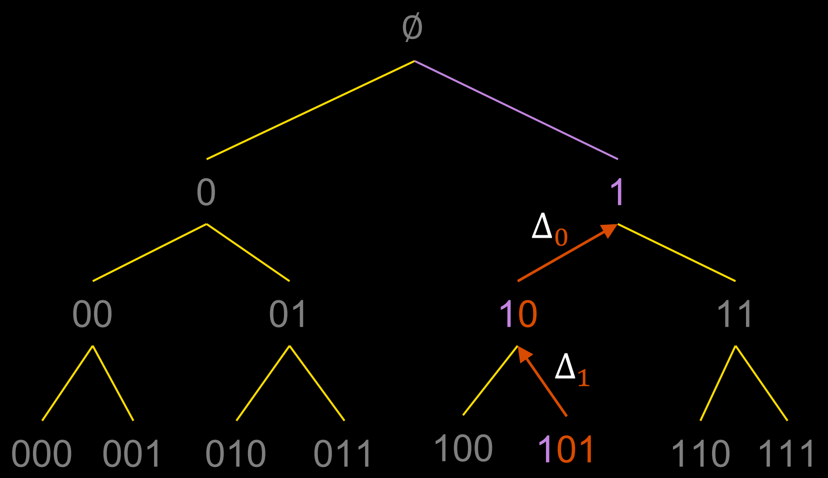
FUNCTIONAL DERIVATIVES OF SIGNATURES

- Recall that $\Delta_0 f = \Delta_t f, \quad \Delta_1 f = \Delta_x f$

- Recursion

    - $\Delta_0 S_{\alpha 0} = S_\alpha, \qquad \Delta_1 S_{\alpha 0} = 0$

    - $\Delta_0 S_{\alpha 1} = 0, \qquad \Delta_1 S_{\alpha 1} = S_\alpha$



$\Longrightarrow$ $\Delta_\gamma S_\alpha \neq 0$ if and only if $\alpha = \beta\gamma$. If so, then $\Delta_\gamma S_\alpha = S_\beta$

In particular, $\Delta_\gamma S_\alpha(X_0) = \delta_{\gamma\alpha}$

SIGNATURE SPANNING

SPACE SPANNED BY ORDER K SIGNATURES

- For a given T, define:
  - A_K, the space spanned by $2^{K+1}-1$ signatures from words of order $\leq K$ (linearly dependent, e.g. $S_{01} = TS_1 - S_{10}$)
  - B_K, the space spanned by 2^K signatures from words of order $= \mathrm{K}$
- Then

$$A_K = B_K$$
$$\dim(A_K) = \dim(B_K) = 2^K$$

- The 2^K words of length K are linearly independent and form a basis of B_K, hence of A_K. However, it's beneficial to have a incremental basis when increasing the order.

INCREMENTAL BASIS

Incremental basis: at step $\mathrm{K}+1$ add 2^{K} words of the 2^{K+1} words of length $\mathrm{K}+1$

There are many (but not all) ways to do it.

Favored choice: at each step take the words that end with 1 (plus ø word).

$$(\emptyset)$$

$$(1)$$

$$(0,1),(1,1)$$

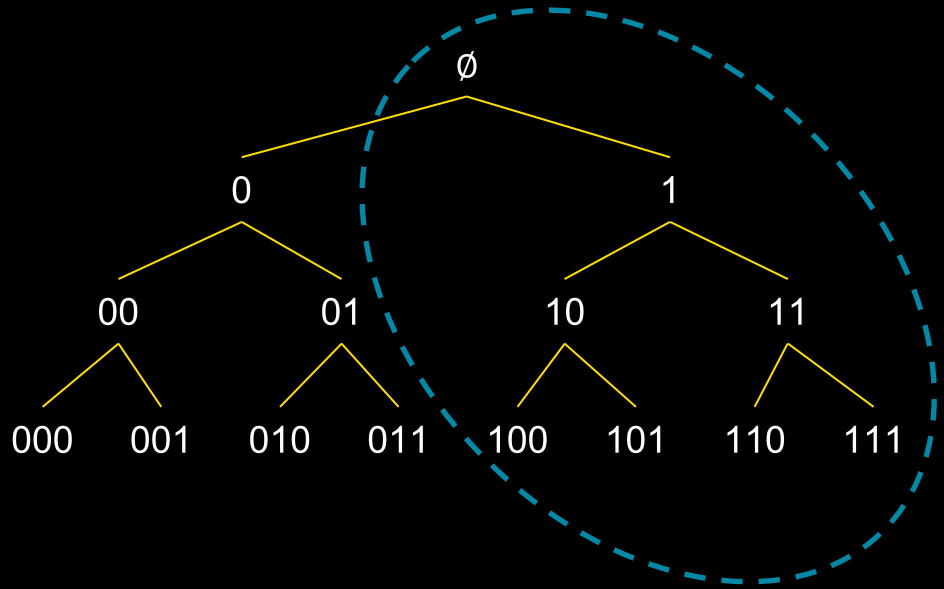
$$(0,0,1),(0,1,1),(1,0,1),(1,1,1)$$

...

It provides an interpretation as dynamic stock trading (up to Itô / Stratonovich).

INCREMENTAL BASIS

Words that start with 1 (plus ∅ word):



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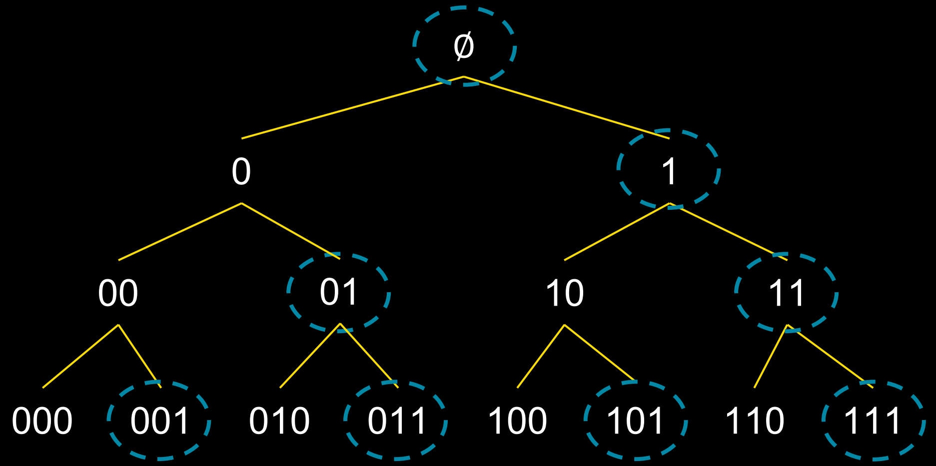
INCREMENTAL BASIS

Words that end with 1 (plus ∅ word):



INDEPENDENT SIGNATURES

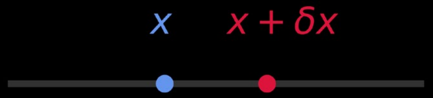
- When considering paths of all lengths (not a fixed T), then all signatures are linearly independent
  - *Proof: If* $f(X_t) = \sum_{|\alpha|<K} c_\alpha \, S_\alpha(X_t) \equiv 0$, *then* $0 = \Delta_\alpha f(X_0) = c_\alpha$ □

- The space generated by signatures of order up to K is of dimension $2^{K+1} - 1$

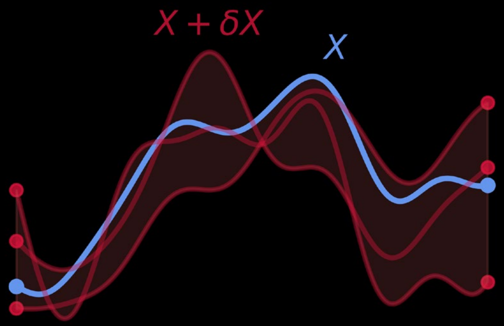
FUNCTIONAL EXPANSIONS

CLASSIFICATION OF EXPANSIONS

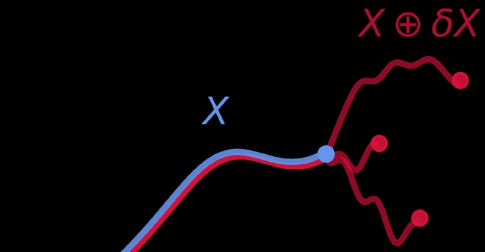
Classical



Static (around a path)



Dynamic (after a path)



TAYLOR AS ITERATED INTEGRALS

Usual 1D case: Applying iteratively the fundamental theorem of calculus:

$$f(x) = f(0) + \int_0^x f'(y)dy$$

$$f'(y) = f'(0) + \int_0^y f''(z)dz$$

$$f''(z) = \cdots$$

leads to the Taylor expansion

$$f(x) = f(0) + \sum_{k=1}^{n} f^{(k)}(0)\frac{x^k}{k!} + \int_0^x f^{(n+1)}(z)\frac{(x-z)^n}{n!}dz$$

The term $\frac{x^k}{k!}$ comes from the iterated integral $\int_0^x \int_0^{x_k} \ldots \int_0^{x_2} dx_1 \ldots dx_{k-1} dx_k$

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WIENER CHAOS EXPANSION

Price functional: $f(X_t) = \mathbb{E}^{\mathbb{Q}}[g(Y_T)|X_t], \quad g: \Lambda_T \to \mathbb{R}$

Martingale Representation: $g(X_T) = \underbrace{\mathbb{E}_0^{\mathbb{Q}}[g(Y_T)]}_{=\phi_0} + \int_0^T \Delta_x f(X_t) dx_t$

Iterate: $\Delta_x f(X_t) = \underbrace{\mathbb{E}_0^{\mathbb{Q}}[\Delta_x f(Y_t)]}_{=\phi_1(t)} + \int_0^t \Delta_x \mathbb{E}^{\mathbb{Q}}[\Delta_x f(Y_t) \mid X_s]\, dx_s$, ...

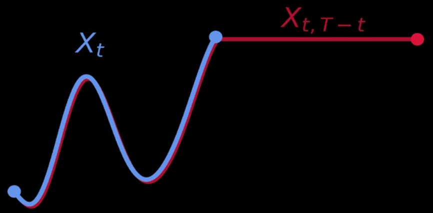
$\Longrightarrow$ **Wiener Chaos**

$$g(X_T) = \phi_0 + \sum_{k \geq 1} \int_{0<t_1<\cdots<t_k<T} \phi_k(t_1, \dots, t_k)\, dx^{\otimes k}$$

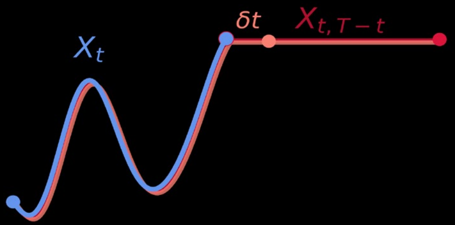
Malliavin calculus: $\phi_k(t_1, \dots, t_k) = \mathbb{E}_0^{\mathbb{Q}}\left[D_{t_1 \cdots t_k} g(Y_T)\right]$

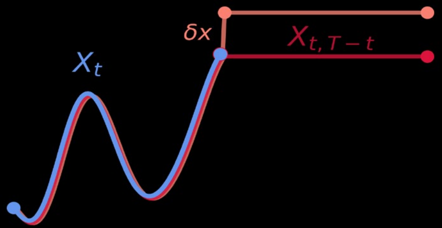
INTRINSIC VALUE FUNCTIONAL

Given a T-functional g, one can associate a functional $f(X_t) = g\left(X_{t,T-t}\right), \quad f(X_T) = g(X_T)$



Properties: (i) $\Delta_t f(X_t) = 0$ (ii) $\Delta_x f(X_t) = D_t g\left(X_{t,T-t}\right)$



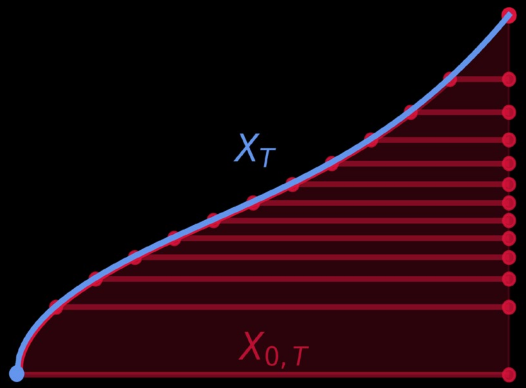


$\Longrightarrow$ Functional Stratonovich: $\mathrm{g}(X_T) = g\left(X_{0,T}\right) + \int_0^T D_t g\left(X_{t,T-t}\right) \circ dx_t$

INTRINSIC VALUE EXPANSION

Stratonovich: $g(X_T) = g(X_{0,T}) + \int_0^T D_t g(X_{t,T-t}) \circ dx_t$

Iterate: $D_t g(X_{t,T-t}) = D_t\, g(X_{0,T}) + \int_0^t D_{st} g(X_{s,T-s}) \circ dx_s, \ldots$



⇒ **Intrinsic Value Expansion**

$$g(X_T) = g(X_{0,T}) + \sum_{k \geq 1} \int_{0<t_1<\cdots<t_k<T} D_{t_1 \cdots t_k} g(X_{0,T}) \circ dx^{\otimes k}$$

TAYLOR EXPANSION OF A FUNCTIONAL

Define $x_{t_i}^0 = t_i \quad x_{t_i}^1 = x_{t_i} \quad dx^\alpha = dx_{t_1}^{\alpha_1} \circ \cdots \circ dx_{t_K}^{\alpha_K}$

$$\Delta_\alpha f = \Delta_{\alpha_1,\dots,\alpha_K} f = \Delta_{\alpha_1}\left(\Delta_{\alpha_2}\left(\dots\left(\Delta_{\alpha_K} f\right)\right)\right)$$

For instance, $\Delta_{01} f = \Delta_0(\Delta_1 f) = \Delta_t(\Delta_x f)$

By induction, one gets the Taylor (Maclaurin) formula:

$$f(X_t) = \sum_{|\alpha|<K} \Delta_\alpha f(X_0)\, S_\alpha(X_t) + r_K(X_t)$$

with the convention $\Delta_\emptyset f(X_0) = f(X_0)$ and

$$r_K(X_t) = \sum_{|\alpha|=K} \int_{0<t_1<\dots<t_K<t} \Delta_\alpha f\left(X_{t_1}\right) \circ dx^\alpha$$

$\{S_\alpha : \alpha \in \text{Words}\}$ form a basis of the space of functionals

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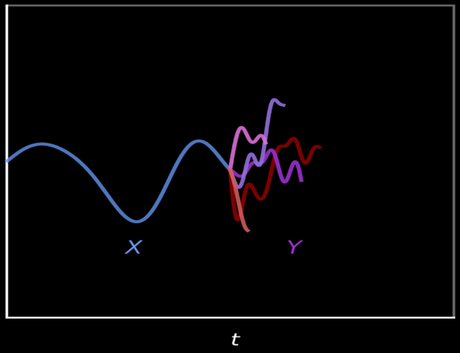
FUNCTIONAL TAYLOR (GENERAL CASE)

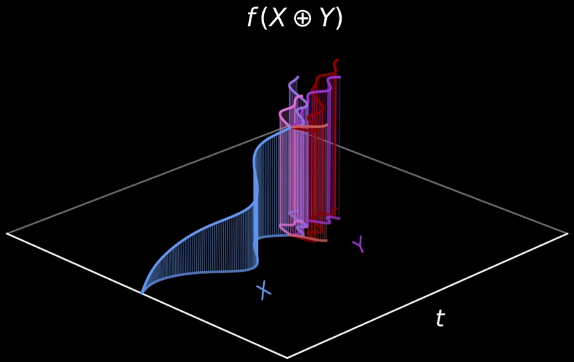
<u>Theorem</u>: Let $X_s \in \Lambda$, $Y_u \in \Omega^{\Pi}$ and f $\in \mathbb{C}^{K,K+1}$. Then,

$$f(X_s \oplus Y_u) = \sum_{|\alpha|<K} \underbrace{\Delta_\alpha f(X_s)}_{\text{functional}} \underbrace{S_\alpha(Y_u)}_{\text{path}} + \underbrace{r_K(X_s, Y_u)}_{\text{remainder}}$$

Sum over words of product of 2 terms

- K^{th} functional derivative at 0 (depending only on the functional, not on the path)
- K-fold integral of path (signature, depending on the path, not on the functional)





2 USES

Functional Maclaurin Expansion

$$f(X_t) = \sum_{|\alpha|<K} \underbrace{\Delta_\alpha f(X_0)}_{\text{functional}} \underbrace{S_\alpha(X_t)}_{\text{path}} + r_K(X_t)$$

1. Option pricing: cubature (expectation)

   - Find “good” weighted paths that price the signatures of the short words correctly

2. Hedging: matching the coefficients (pathwise)

   - Find a hedge with the same first coefficients

SUMMARY

- **Wiener Chaos**

$$g(X_T) = \mathbb{E}^{\mathbb{Q}}[g(Y_T)] + \sum_{k < K} \int_{0<t_1<\cdots<t_k<T} \mathbb{E}^{\mathbb{Q}}\left[D_{t_1\cdots t_k} g(Y_T)\right] dx^{\otimes k} + r_K^{WC}$$

- **Intrinsic Value**

$$g(X_T) = g\left(X_{0,T}\right) + \sum_{k < K} \int_{0<t_1<\cdots<t_k<T} D_{t_1\cdots t_k} g(X_{0,T}) \circ dx^{\otimes k} + r_K^{IV}$$

- **Functional Taylor**

$$f(X_t) = f(X_0) + \sum_{|\alpha| < K} \Delta_\alpha f(X_0)\, S_\alpha(X_t) + r_K^{FT}$$

LINKING INTRINSIC AND FUNCTIONAL EXPANSIONS

IV expansion

$$g(X_T) = g\left(X_{0,T}\right) + \sum_{k < K} \int_{0<t_1<\cdots<t_k<T} D_{t_1\cdots t_k} g(X_{0,T}) \circ dx^{\otimes k} + r_K^{IV}$$

Functional expansion

$$g(X_T) = g\left(X_{0,T}\right) + \sum_{|\alpha| < K} \Delta_\alpha f(X_0)\, S_\alpha(X_T) + r_K^{FT}$$

We infer that

$$D_{t_1\cdots t_k} g\left(X_{0,T}\right) = \sum_{|\alpha|_1 = k} \Delta_\alpha f(X_0) \prod_{l=0}^{k} \frac{(t_{l+1} - t_l)^{\gamma_l}}{\gamma_l!}$$

Where γ_l is the number of 0's between 2 consecutive 1's

FROM FUNCTIONAL EXPANSION TO T-FUNCTIONAL EXPANSION

We have seen the expansion of a functional. What about the expansion of a T-functional, $g(X_T)$?

- Find a functional $f(X_t)$ such that $f(X_T) = g(X_T)$ **(smooth embedding)**
- Expand f: $f(X_t) = \sum_\alpha \Delta_\alpha f(X_0)\, S_\alpha(X_t)$
- Evaluate at X_T: $g(X_T) = f(X_T) = \sum_\alpha \Delta_\alpha f(X_0)\, S_\alpha(X_T)$
- Rewrite in the incremental basis $g(X_T) = \lambda_\emptyset + \sum_\alpha \lambda_\alpha\, S_{(\alpha 1)}(X_T)$
- The decomposition does not depend on the choice of f

APPLICATIONS

EUROPEAN AND EXOTIC OPTIONS

- European options are spanned by the words $(1,1..,1)$

$$S_{(1,\ldots,1)_k}(X_T) = \frac{x_T^k}{k!}$$

It gives the moments of x_T.

- Exotic options are spanned by all the words

Iterated integrals $S_\alpha(X_T)$ are the building blocks of path dependence.

EUROPEAN OPTION EXAMPLE

Approximation of a T-maturity European call by a linear combination of T-signature payoffs

Payoff at T: $(x_T - x_0)^+$

$$S_1(X_T) = x_T$$

$$S_{11}(X_T) = x_T^2$$

$$\dots$$

$$S_{\underbrace{1\dots1}_{k}}(X_T) = \frac{x_T^k}{k!}$$

FORWARD T1<T EXAMPLE

Approximation of a T_1-maturity forward by a linear combination of T-signature payoffs, $T_1 < T$

Payoff at T:

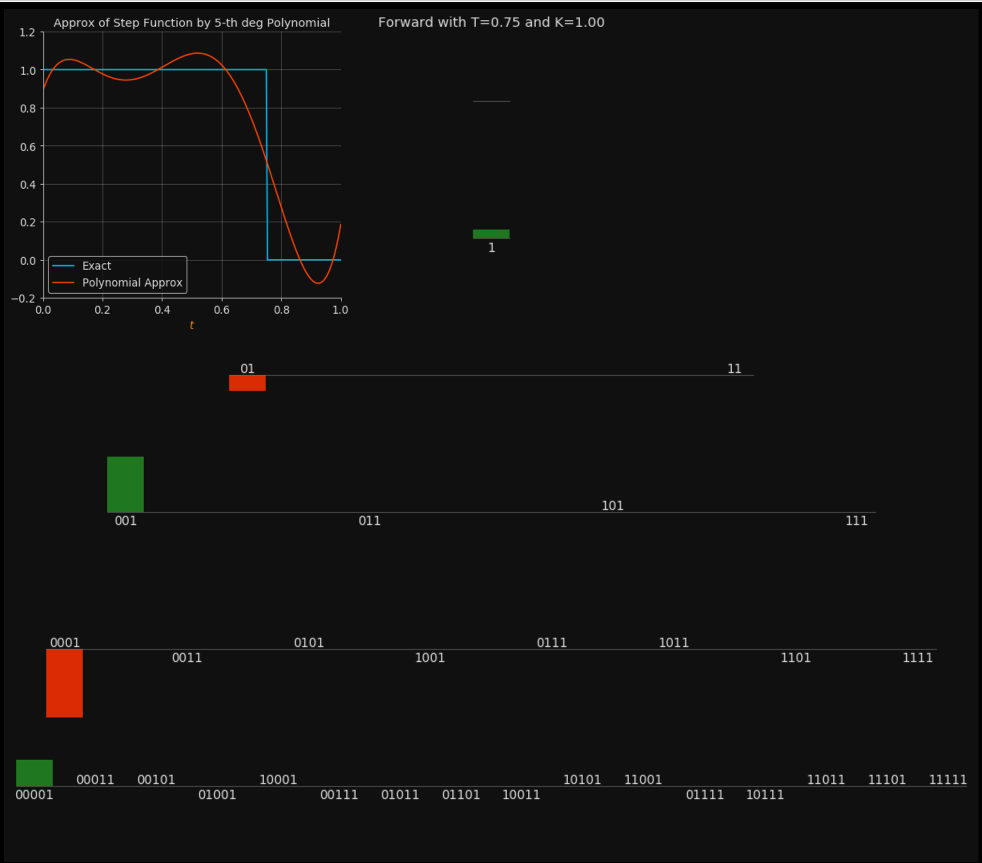
$$x_{T_1} - x_0 = \int_0^{T_1} dx_t = \int_0^T H(T_1 - t)dx_t$$

$$S_1(X_T) = \int_0^T 1\, dx_t$$

$$S_{01}(X_T) = \int_0^T t\, dx_t$$

...

$$S_{\underbrace{0\cdots0}_{k}1}(X_T) = \int_0^T \frac{t^k}{k!}\, dx_t$$



FORWARD START EXAMPLE

Approximation of a $T_1 \times T_2$ forward-start call by a linear combination of T-signature payoffs, $T_1 < T_2 < T$

Payoff at T:

$$(x_{T_2} - x_{T_1})^+$$
$$= \left(\int_0^T [H(T_2 - t) - H(T_1 - t)] dx_t \right)^+$$

DEEP HEDGE MADE SIMPLE

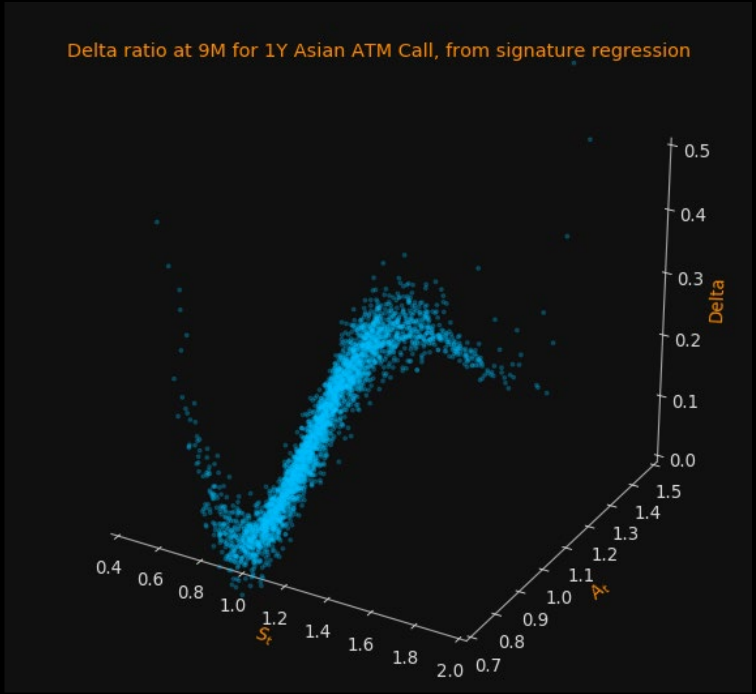
Given a set of paths, historical or synthetic, and a derivative payoff, find the best delta hedge $h(X_t)$, as a functional of X_t, the path so far.

- Neural net approach: parameterize $h(X_t)$ by a neural network (LSTM, GRU, …); train the network by gradient descent, or

- Signature approach: write $h(X_t)$ as a weighted sum of signatures:

$$h(X_t) = \sum_{\alpha} \lambda_\alpha S_\alpha(X_t)$$

Find the weights λ_α by multiple regression.



CONCLUSION

- Signatures are natural building blocks of path-dependent options
- In the (t, x_t) case, we show a reconstruction of the path from its signature using Legendre polynomials
- For continuous paths with finite quadratic variation along a given sequence of partitions, we establish
  - A pathwise functional Taylor expansion
  - An expansion for T –functionals based on the intrinsic value
- Comparison with classical Taylor expansion and Wiener chaos
- The functional Taylor expansion decomposes exotic claims. This opens the door to novel pricing and hedging algorithms

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Thank You