Stochastic flows and rough differential equations on foliated spaces

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Aim of the talk

- We consider SDE on compact foliated space. First introduced and solved by Suzaki (2015)
- We prove that stochastic flow associted to it exists.
- Our method is rough path theory, because Kolmogorov-Čentsov continuity criterion is UNavailable.
- From a viewpoint of RP theory, there is no big difficulty in constructing the flow.
- Our work may open the door for full stochastic analysis on foliated spaces (SDE theory, rough path theory, Malliavin calculus, path space analysis, etc.)

Consider the following SDE on $\mathbb{R}^n$ or manifold:

$$dx_t = \sum_{i=1}^{d} V_i(x_t) \circ dw_t^i + V_0(x_t)dt, \quad x_0 = \xi \text{ (deterministic)}.$$

Here, V_i's are nice vector fields, $(w_t)_{t\geq 0}$ is d-dim BM, ξ is an initial value. We often write $x_t = x(t, \xi, w)$.

• $\xi \mapsto x(t, \xi, w)$ is called the stochastic flow of homeo/diffeomorphism associated with the SDE.
• Stochastic flows play key roles in stochastic analysis over (Riemannian) manifolds. ($\exists$ Many deep resluts.)

One of the hardest parts of stochastic flow theory is its existence, i.e., the existence of a r.v.

$$w \mapsto [\xi \mapsto x(t, \xi, w)]$$

because the negligible null set for the SDE depends on ξ and there are uncountably many ξ's.
The standard (and only?) tool to overcome this difficulty is Kolmogorov-Čentsov criterion for $\exists$ of conti. modification.

$$\mathbf{E}[|x(t,\xi,\cdot) - x(s,\eta,\cdot)|^{\heartsuit}] \lesssim \mathrm{dist}\,((t,\xi),(s,\eta))^{n+1+\spadesuit}$$

But, this criterion (basically) works only on (a subset of) Euclidean space.

- Let $\mathcal{M}$ be a compact foliated space. $\mathcal{M}$ itself and its transversal direction are just (locally compact) metric spaces.
- But, a certain differential structure is given ("leafwise C^k").
- So, there are SDEs on $\mathcal{M}$:

$$dx_t = \sum_{i=1}^{d} V_i(x_t) \circ dw_t^i + V_0(x_t)dt, \quad x_0 = \xi \text{ (deterministic)}.$$

Here, V_i's are leafwise smooth (or C^3) vector fields, $(w_t)_{t \geq 0}$ is d-dim BM, $\xi \in \mathcal{M}$ is an initial value.

Formulated and solved by Suzaki ('15) for every fixed ξ.
But, KC criterion is NOT available (even when $\mathcal{M}$ happens to have a manifold structure). $\exists$ Stochastic Flow?

- To prove the existence of $w \mapsto [\xi \mapsto x(t, \xi, w)]$, we will use Rough Path Theory.
- RP theory is a "deterministic version" of Itô's SDE theory.
- The solution map of rough differential eq., Lyons-Itô map, is continuous in all input data (ξ, V_i, "the lift of w").
- RDE naturally generates a flow in a deterministic way.
- Only probabilistic part is lifting the noise $w \mapsto W$ (BRP). Hence, this is the only place where "exceptinal null set" appears. Notice it is clearly independent of ξ.

Quite natural to guess: If we define RDE on $\mathcal{M}$, then we can easily construct the stochastic flow on $\mathcal{M}$. ♡♡
(Loosely, this is our main result.)

Rough Differential Equation

- Geometric Rough Path

$\triangle := \{(s,t) \mid 0 \le s \le t \le 1\}, \quad \alpha \in (0,1]$,
$A : \triangle \to \mathbb{R}^d$, conti.

$$\|A\|_\alpha := \sup_{0 \le s < t \le 1} |A_{s,t}| / |t-s|^\alpha$$

$T^{(2)}(\mathbb{R}^d) := \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d)$
(truncated tensor algebra of step 2)

Definition (rough path) $\alpha \in (1/3, 1/2]$ "roughness"
A conti. map $\mathbf{W} = (1, W^1, W^2) : \triangle \to T^{(2)}(\mathbb{R}^d)$
is said to be a rough path if

(i) K. T. Chen's identity $0 \le s \le u \le t \le 1$,

$$\begin{aligned} W^1_{s,t} &= W^1_{s,u} + W^1_{u,t}, \\ W^2_{s,t} &= W^2_{s,u} + W^2_{u,t} + W^1_{s,u} \otimes W^1_{u,t} \end{aligned}$$

(ii) α-Hölder condition $\|W^1\|_\alpha < \infty, \ \|W^2\|_{2\alpha} < \infty.$ -

$\Omega_\alpha(\mathbb{R}^d)$: The set of α-Hölder RPs. $2 = [1/\alpha]$
(We will write $\mathbf{W} = (W^1, W^2)$ for simplicity.)

Example (smooth RP)

$h : [0, 1] \to \mathbb{R}^d$; a Camerom-Martin path.

$$H^1_{s,t} := h_t - h_s, \quad H^2_{s,t} := \int_s^t (h_u - h_s) \otimes dh_u$$

This $\mathbf{H} = (H^1, H^2)$ is clearly a RP. Lift of h.

The lift map is denoted by $\mathcal{L}$, i.e., $\mathbf{H} = \mathcal{L}(h)$.

Definition (the geometric RP space) (complete, separable)

$$G\Omega_\alpha(\mathbb{R}^d) := \overline{\{\mathcal{L}(h) \mid h \in \mathcal{H}\}}^{d_\alpha} \subset \Omega_\alpha(\mathbb{R}^d).$$

Rough Differential Equation on $\mathbb{R}^n$

$V_i\colon \mathbb{R}^n \to \mathbb{R}^n$, C_b^3. $(0 \le i \le d)$
(Often viewed as vector fields on $\mathbb{R}^n$.)

- For $\mathbf{W} = (W^1, W^2) \in G\Omega_\alpha(\mathbb{R}^d)$, consider the following equaiton on $\mathbb{R}^n$. This is called RDE driven by $\mathbf{W}$:

$$dx_t = \sum_{i=1}^{d} V_i(x_t) d\mathbf{W}_t^i + V_0(x_t) dt, \quad x_0 = \xi \in \mathbb{R}^n$$

(The superscript i denotes the coordinate, not the level.)

• If $\mathbf{W}$ is the natural lift of Brownian motion $(w_t)_{t\geq 0}$, then RDE solution coincides with the solution of usual Stratonovich SDE a.s.:

$$dx_t = \sum_{i=1}^{d} V_i(x_t) \circ dw_t^i + V_0(x_t)dt, \quad x_0 = \xi$$

• If $\mathbf{W}$ is a natural lift of a Cameron-Martin path h, i.e., $\mathbf{W} = \mathcal{L}(h)$, then RDE solution coincides with the solution of

$$dx_t = \sum_{i=1}^{d} V_i(x_t) dh_t^i + V_0(x_t)dt, \quad x_0 = \xi$$

in the standard sense.

Lyons' continuity theorem

The (deterministic) solution map

$$(\mathbf{W}, \xi, \{V_i\}) \quad \mapsto \quad x \in C^\alpha([0,1], \mathbb{R}^n)$$

is continuous from $G\Omega_\alpha(\mathbb{R}^d) \times \mathbb{R}^n \times C_b^3(\mathbb{R}^n, \mathbb{R}^n)^{d+1}$.

[Remark] For our porpose, continuity in $\{V_i\}$ is crucial.
[Remark] Though the definition of geometric RP is unique, $\exists$ several formulations of RDE (at least 6 or 7?).

Brownian Rough Path

$w = (w_t)_{0 \le t \le 1}$: d-dim BM.
$w(m) = (w(m)_t)_{0 \le t \le 1}$: dyadic piecewise linear approximation of w associated with the partition $\{j/2^m \colon 0 \le j \le 2^m\}$.

Then, the following set is of Wiener measure 1:

$$\big\{w \colon \ \{\mathcal{L}(w(m))\}_{m=1}^{\infty} \text{ is Cauchy in } G\Omega_\alpha(\mathbb{R}^d)\big\}$$

So, we set $\mathcal{L}(w) = \lim_{m\to\infty} \mathcal{L}(w(m))$ if w belongs to the above subset (and set $\mathcal{L}(w)$ to be zero-RP if otherwise).

Then, $\mathcal{L}\colon C_0([0,1],\mathbb{R}^d) \to G\Omega_\alpha(\mathbb{R}^d)$ is a everywhere-defined Borel measurable map.

If we put $\mathbf{W} = \mathcal{L}(w)$ in Lyons-Itô map, then $x = x(\mathbf{W}, \xi, \{V_i\})$ coincides with the solution of corresponding Stratonovich SDE.
(Thanks to Wong-Zakai's approximation & Lyons continuity theirem)

Thus, the solution of SDE is expressed as the image of a continuous map.

Three Major formalisms of RDE

- Lyons' original formulation
  Solution is a fixed point of rough integral equation.
  Both RP integrals and solutions are rough paths.
- Gubinelli's formulation
  Solution is a fixed point of rough integral equation.
  Both RP integrals and solutions are controlled paths
  w.r.t. a given rough path $\mathbf{W} \in G\Omega_\alpha(\mathbb{R}^d)$.
- Davie's formulation Use Euler-Taylor type expansion as definition. Solution is a usual path. ($\nexists$ integral eq.)
  $\exists$ some variants, e.g., Bailleul's works.

One of the variants of Davie's formulation
(by Bailleul '15. See also Cass-Wiedner '16+):
$(x_t)_{0\le t\le 1}$ solves the RDE driven by $\mathbf{W} = (W^1, W^2)$ if and only if

$$f(x_t) - f(x_s) = \sum_{i=1}^{d} V_i f(x_s) W^{1,i}_{s,t} + \sum_{j,k=1}^{d} V_j V_k f(x_s) W^{2,jk}_{s,t} + V_0 f(x_s)(t-s) + O(|t-s|^{3\alpha}), \quad \forall f \in C^3(\mathbb{R}^n, \mathbb{R}).$$

This formulation works very well on manifolds because

- A solution is a usual path (No "higher objects").
- Independent of the choice of local chart.

So we will use this type of formulation.

Foliated Spaces

- Let $\mathcal{M}$, $\mathcal{Z}$ be locally compact metric space.
- Let $A \subset \mathbb{R}^p$, $B \subset \mathcal{Z}$ be open. A function $f : A \times B \to \mathbb{R}^n$ is called <u>leafwise C^k</u> if $f = f(y, z)$ is C^k in y for each fixed z and the derivatives are continous in (y, z).

- Let $\phi : A \times B \to \hat{A} \times \hat{B}$ is called <u>leafwise C^k</u> if it is of the form $\phi(y, z) = (f(y, z), g(z))$ for some $f \in C^k_L$ and some continuous g.

Definition (foliated space) $\mathcal{M}$ is called a p-dimensional foliated space (transversely modelled on $\mathcal{Z}$) if the following conditions are satisfied:

- $\exists$ open cover $\{U_\beta\}$ of $\mathcal{M}$, $\exists$ homeo $\phi_\beta \colon U_\beta \to A_\beta \times B_\beta$, where $A_\beta \subset \mathbb{R}^p$, $B_\beta \subset \mathcal{Z}$ are certain open subsets.
- $\phi_\beta \circ \phi_\gamma^{-1} \colon \phi_\gamma(U_\beta \cap U_\gamma) \to \phi_\beta(U_\beta \cap U_\gamma)$ are leafwise C^∞.

- A set of the form $\phi_\beta(A_\beta \times \{z\})$ is called a plaque.
- Patching together intersecting plaques, you get a leaf on $\mathcal{M}$.
- Each leaf is a C^∞-manifold. Different leaves never intersect.

♣ Foliated manifold $\Longrightarrow$ Lamination $\Longrightarrow$ Foliated space.
♠ In what follows, $\mathcal{M}$ is assumed to be compact.

Example (Mapping Torus)

- $\mathcal{Z}$: compact metric space. $F: \mathcal{Z} \to \mathcal{Z}$: homeomorphism.
- $\mathbb{Z}$-action on $\mathbb{R} \times \mathcal{Z}$:

$$k \cdot (y, z) = (y + k, F^k(z)), \qquad k \in \mathbb{Z}.$$

- Then, the quotient space $\mathcal{M} := \mathbb{R} \times \mathcal{Z}/\mathbb{Z}$ is a compact 1-dim FS modelled transversally on $\mathcal{Z}$.

RDE/SDE on Foliated Spaces

Let V_i $(1 \le i \le d)$ be leafwise C^3 vector fields. SDEs on $\mathcal{M}$:

$$dx_t = \sum_{i=1}^{d} V_i(x_t) \circ dw_t^i + V_0(x_t)dt, \quad x_0 = \xi \text{ (determinstic)}.$$

Formulated and solved by Suzaki (2015).

The corresponding RDE should be

$$dx_t = \sum_{i=1}^{d} V_i(x_t) d\mathbf{W}_t^i + V_0(x_t)dt, \quad x_0 = \xi \text{ (determinstic)}.$$

Definition (Solution to RDE)

$(x_t)_{0 \le t \le 1}$ is said to solve the RDE on $\mathcal{M}$ if $x_0 = \xi$ and

$$f(x_t) - f(x_s) = \sum_{i=1}^{d} V_i f(x_s) W_{s,t}^{1,i} + \sum_{j,k=1}^{d} V_j V_k f(x_s) W_{s,t}^{2,jk} + V_0 f(x_s)(t-s) + O(|t-s|^{3\alpha}), \quad \forall f \in C_L^3(\mathcal{M})$$

[Remark]

A (time-local) solution never gets out of the initial plaque.
Hence, a solution stays in one leaf.

[Fact]

$\exists !$ unique global solution for every ξ and $\mathbf{W} = (W^1, W^2)$.

[Key Point]

In a local chart

$$\mathcal{M} \supset U \ni \quad x \longleftrightarrow (y, z) \quad \in A \times B \subset \mathbb{R}^p \times \mathcal{Z},$$

the RDE on $\mathcal{M}$ is equivalent to the following one on $\mathbb{R}^p$:

$$dy_t = \sum_{i=1}^{d} V_i(y_t, z_0) d\mathbf{W}_t^i + V_0(y_t, z_0) dt, \qquad \phi(\xi) = (y_0, z_0)$$

Therefore, varying the initial value ξ in the transversal direction amounts to varying the coefficient vector fields on $\mathbb{R}^p$-valued RDE. (The continuity in ξ is heuristically evident.)

$\Longrightarrow$ The flow associated with RDE on $\mathcal{M}$ exists and it is a "leafwise homeomorphism"

Main Result

Theorem 1 (I.-Suzaki, '20)

Let $\mathbf{W}$ *be Brownian rough path and consider the RDE on* $\mathcal{M}$ *driven by* $\mathbf{W}$*. Then, the global solution* $x_t = x(t, \xi, \mathbf{W})$ *coincides with the solution of corresponding stratonovich SDE. Moreover,*

$$w \quad \mapsto \quad \big[(t, \xi) \mapsto x(t, \xi, \mathbf{W}) \in \mathcal{M}\big]$$

almost surely defines a flow of leafwise homeomorphisms.

- In reality, this is a flow of leaf-preserving leafwise diffeomorphisms of $\mathcal{M}$.

Some comments are in order:

- The only exceptional null set is $\{w \colon w \text{ does not admit a RP lift}\}$. But, this is clearly independent of the initial value ξ.
- The inverse flow is given by the solution to the RDE driven by the time reversal of the same rough path.

Simple Applications

[1] As usual, the heat semigroup associated with $\frac{1}{2}\sum_{i=1}^{d} V_i^2 + V_0$ admits a Feynman-Kac representation:

$$T_t f(\xi) = \mathbb{E}[f(x(t,\xi,\mathbf{W}))].$$

Suzaki (2015) showed Feller property, i.e., $f \in C(\mathcal{M}) \Longrightarrow T_t f \in C(\mathcal{M})$ by checking the continuity $\xi \mapsto x(\bullet,\xi,w)$ in the sense of limit in probability. His proof is rather long.

Now this fact immediately follows from our main result.

[2] Measurability issue: In Suzaki (2015),

$$(\xi, w) \mapsto x(\bullet, \xi, w) \qquad \text{(strong sol. of SDE)}$$

is only shown to be measurable w.r.t.

$$\bigcap \{ \overline{\mathcal{B}(\mathcal{M}) \otimes \mathcal{B}(C_0([0,1], \mathbb{R}^d))}^{m \times \mu} : m \in \mathrm{Prob}(\mathcal{M}) \},$$

where μ is the Wiener measure.

But, this σ-field looks a bit too large.

In our approach, it is written as the composition of Lyons-Itô map and RP lift $\mathcal{L}$ (i.e. $\mathbf{W} = \mathcal{L}(w)$).

$$(\xi, w) \mapsto x(\bullet, \xi, w) = x(\bullet, \xi, \mathcal{L}(w)).$$

So, as an everywhere defined map, this is measurable w.r.t.

$$\mathcal{B}(\mathcal{M}) \otimes \mathcal{B}(C_0([0,1], \mathbb{R}^d)).$$

As a μ-equivalence class, this is measurable w.r.t.

$$\mathcal{B}(\mathcal{M}) \otimes \overline{\mathcal{B}(C_0([0,1], \mathbb{R}^d))}^{\mu}.$$

Therefore, we have slightly improved the previous work.

The End