Variational principles for fluid dynamics on rough paths

James-Michael Leahy

Department of Mathematics at Imperial College London
and Applied Mathematics at University of Twente
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Outline

Lagrangian mechanics

- Let Q be a smooth configuration manifold and $L \in C^1(TQ;\mathbb{R})$.

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- Define the action functional $S : \Omega(q_1, q_2) \to \mathbb{R}$ by

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where $\Omega(q_1, q_2) = \left\{q \in C^2([a,b];Q) : q_a = q_1 \text{ and } q_b = q_2\right\}.$

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- We say $q \in \Omega(q_1, q_2)$ is a critical point of the action functional if for all 'variations' of q, that is, $\delta q = [c.]_q \in T_q\Omega(q_1, q_2)$,

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} S(c_\epsilon) = \mathbf{d}S(q) \cdot \delta q = 0.$$

In a local trivilialization chart of TQ,

$$c_\epsilon(t) = q_t + \epsilon \delta q_t, \quad \text{with } \delta q \in C^2([a,b]; Q), \ \delta q_a = \delta q_b = 0.$$

Hamilton's principle

Theorem

A curve q is a critical point of S iff in a local trivialization chart of TQ

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}^i}(q,\dot{q})\right] = \frac{\partial L}{\partial q^i}(q,\dot{q}).$$

These equations are called the Euler-Lagrange equations. They are a system of second-order ODEs if $\frac{\partial L}{\partial \dot{q}^j \partial \dot{q}^i}$ has non-vanishing determinant.

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Proof.

Integrating by parts and using that $\delta q_a = \delta q_b = 0$, we find

$$\begin{aligned}\mathbf{d}S \cdot \delta q &= \int_a^b \left(\frac{\partial L}{\partial q^i}(q,\dot{q})\delta q^i + \frac{\partial L}{\partial \dot{q}^i}(q,\dot{q})\frac{\mathrm{d}}{\mathrm{d}t}\delta q^i\right)\mathrm{d}t \\ &= \int_a^b \left(\frac{\partial L}{\partial q^i}(q,\dot{q}) - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}^i}(q,\dot{q})\right]\right)\delta q^i \,\mathrm{d}t.\end{aligned}$$

□

Newtonian and Hamiltonian systems

- A Newtonian potential system of N point masses in $\mathbb{R}^d$ is equivalent to the Euler-Lagrange equations with $Q = \mathbb{R}^{dN}$, $TQ = \mathbb{R}^{2dN}$, and

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{n=1}^{N} m_n \frac{1}{2} |\dot{\mathbf{q}}_n|^2 - V(\mathbf{q}).$$

Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\mathbf{q}}^n}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial L}{\partial \mathbf{q}_n}(\mathbf{q}, \dot{\mathbf{q}}) \quad \Leftrightarrow \quad m_n \ddot{\mathbf{q}}_n(t) = -\frac{\partial V}{\partial \mathbf{q}_n}(\mathbf{q}_t)$$

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- One passes to Hamiltonian dynamics via the Legendre transformation to get

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}},$$

  where

$$H(\mathbf{p}, \mathbf{q}) = \sup_{\mathbf{q}} (\mathbf{p} \cdot \dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}})) = \sum_{n=1}^{N} \frac{1}{2m_n} |\dot{\mathbf{p}}_n|^2 + V(\mathbf{q}).$$

Geodesic equation

- Let (M, g) be a Riemannian manifold with volume form μ_g.
- Define $S : \Omega(q_1, q_2) \to \mathbb{R}_+$ by

$$S(q) = E(q) = \int_a^b L(\dot{q}_t)\mathrm{d}t = \frac{1}{2}\int_a^b g_{q_t}(\dot{q}_t, \dot{q}_t)\mathrm{d}t.$$

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- The Euler-Lagrange equation is the geodesic equation

$$\frac{\mathrm{d}^2 q^a}{\mathrm{d}t^2} + \Gamma^q_{bc}\frac{\mathrm{d}q^b}{\mathrm{d}t}\frac{\mathrm{d}q^c}{\mathrm{d}t} = 0,$$

where Γ is the Christoffel symbol. Alternatively,

$$\nabla_{\dot{q}}\dot{q} = 0,$$

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  where Γ is the Christoffel symbol. Alternatively,

$$\nabla_{\dot{q}}\dot{q} = 0,$$

  where ∇ is the Levi-Civita connection.
- Geodesics are not always global minimizers of the energy functional, but they are local minimizers.

Lie groups

- Let G be a Lie group with identity e and Lie algebra $\mathfrak{g} = T_e G$. For example, (G = GL(d), $\mathfrak{g}$ = Mat(d)) or (G = SO(d),$\mathfrak{g} = \mathfrak{so}(d)$).

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- Assume that the Lagrangian is right-invariant under the group action:

$$L(g, \dot{g}) = L(e, \dot{g}g^{-1}) =: \ell(u), \quad u = \dot{g}g^{-1} \in \mathfrak{g}.$$

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- Assume that $\frac{\delta \ell}{\delta u} : \mathfrak{g} \to \mathfrak{g}^*$ is a diffeomorphism, where

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \ell(u + \epsilon \delta u) = \langle \frac{\delta \ell}{\delta u}(u), \delta u \rangle_{\mathfrak{g}} \quad \forall\ \delta u \in \mathfrak{g}.$$

Reducing variations to the Lie algebra

Let $g \in \Omega(g_1, g_2)$ and $\delta g = [c.]_g \in T_g\Omega(g_1, g_2)$. Set

$$u = \dot{g}g^{-1} \in C^1([a,b];\mathfrak{g}), \quad \delta w = \dot{\delta g}g^{-1} \in C^1([a,b];\mathfrak{g}),$$

and

$$\delta u_t = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \dot{c}_\epsilon(t)c_\epsilon^{-1}(t) \in C^1([a,b];\mathfrak{g}).$$

Lemma

If

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\epsilon}c_\epsilon(t) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\frac{\mathrm{d}}{\mathrm{d}t}c_\epsilon(t), \tag{1}$$

then

$$\delta u_t = \dot{\delta w}_t - \mathrm{ad}_{u_t}\,\delta w_t = \dot{\delta w}_t + [u_t, \delta w_t].$$

Euler-Poincaré reduction

Theorem

For a curve $g \in \Omega(g_1, g_2)$ with $u = \dot{g}g^{-1} \in C^1([a,b];\mathfrak{g})$, TFAE

- *g satisfies the Euler-Lagrange equations;*
- *g is a critical point of $S(g) = \int_a^b L(g_t, \dot{g}_t)\,\mathrm{d}t$;*
- *u satisfies the Euler-Poincaré equations:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta \ell}{\delta u} + \mathrm{ad}_u^* \frac{\delta \ell}{\delta u} = 0;$$

- *$(g, u, \lambda = \frac{\delta \ell}{\delta u}(u))$ is a critical point of*

$$S(g, u, \lambda) = \int_a^b \ell(u_t) + \langle \lambda_t, \dot{g}_t g_t^{-1} - u_t \rangle_{\mathfrak{g}};$$

- *u is a critical point of $S(u) = \int_a^b \ell(u_t)\,\mathrm{d}t$ with variations of the form*

$$\delta u_t = \dot{\delta w}(t) - \mathrm{ad}_{u_t} \delta w(t).$$

[Arnold, 1966] topological hydrodynamics

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- We endow $\mathrm{Diff}^s_{\mu_g}$ with the right-invariant (weak/not-complete) metric

$$\langle U, V\rangle_\eta = \int_M g_{\eta(m)}(U(m), V(m))\mu_g(m) = \langle U\eta^{-1}, V\eta^{-1}\rangle_{\mathfrak{g}},$$

where the right-hand-side is the L^2-inner product on vector fields

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- [Ebin and Marsden, 1970] showed there exists a smooth Levi-Civita connection $\bar{\nabla} = P\nabla$ (where $P : \mathfrak{X}^s \to \mathfrak{X}^s_{\mathrm{div}}$) and geodesic spray:

$$\overset{\text{Euler-Lagrange}}{P\nabla_{\dot\eta}\dot\eta = 0} \quad \overset{u=\dot\eta\eta^{-1}}{\Leftrightarrow} \quad \overset{\text{Euler-Poincare}}{\partial_t u + \nabla_u u = -\nabla p.}$$

Parametrization through Euler-Poincare [Holm, 2015]

- Assume a decomposition of the form

$$\dot{g}_t = u_t g_t + \sum_{k=1}^{K} \xi_k g_t \dot{z}_t^k,$$

where u models coarse-scales and $\sum_k \xi_k \dot{z}^k$ models fast-scales.

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- Require that $(g, u, \lambda = \frac{\delta \ell}{\delta u}(u))$ is a critical point of

$$S(g, u, \lambda) = \int_0^T \ell(u_t) + \langle \lambda_t, \dot{g}_t g_t^{-1} - u_t - \sum_{k=1}^{K} \xi_k \dot{z}_t^k \rangle_{\mathfrak{g}}$$

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to derive an equation for u.

- Preserves geometric structure for momentum $\frac{\delta \ell}{\delta u}$.
- Equivalently, u satisfies the Euler-Poincaré equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\delta \ell}{\delta u}\right] + \mathrm{ad}_u^* \frac{\delta \ell}{\delta u} + \sum_{k=1}^{K} \left(\mathrm{ad}_{\xi_k}^* \frac{\delta \ell}{\delta u}\right) \dot{z}_t^k = 0.$$

Stochastic advection by Lie transport [Holm, 2015]

Letting $G = \mathrm{Diff}^s_\mu(\mathbb{T}^d)$, $\mathfrak{g} = \mathfrak{X}^s_{\mathrm{div}}(\mathbb{T}^d)$, and

$$\ell(u) = \int_{\mathbb{T}^d} |u(x)|^2 \mu(x)$$

with $z^k_t = B^k_t$, $1 \le k \le K$, independent Brownian motions, we find

$$\mathrm{d}u^i + u^j \partial_{x^j} u^i \mathrm{d}t + \sum_{k=1}^{K} \left(\xi^j \partial_{x^j} u^j + u^j \partial_{x^i} \xi^j \right) \circ \mathrm{d}B^k_t = -\partial_{x^i} \mathrm{d}p.$$

in the standard coordinate system with $g_{ij} = \delta_{ij}$.

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in the standard coordinate system with $g_{ij} = \delta_{ij}$.

If we denote by $u^\flat$ the one-form associated with u, we find

$$\mathrm{d}u^\flat + \pounds_u u^\flat \mathrm{d}t + \pounds_\xi u^\flat \circ \mathrm{d}B_t = -\mathbf{d}\mathrm{d}p,$$

and hence the vorticity two-form $\omega = \mathbf{d}u^\flat$ satisfies

$$\mathrm{d}\omega + \pounds_u \omega \mathrm{d}t + \pounds_\xi \omega \circ \mathrm{d}B_t = 0.$$

Definition

For a given $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, we say

$$\mathbf{Z} = (z, \mathbb{Z}) \in C_T^{\alpha}(\mathbb{R}^K) \times C_{2,T}^{2\alpha}(\mathbb{R}^{K\times K})$$

is $\mathbb{R}^K$-valued α-Hölder geometric rough path if there exists

$$\mathbf{Z}^{(N)} = \left(z_t^{(N)}, \mathbb{Z}_{st}^{(N)} := \int_s^t \int_s^r \mathrm{d}z_{r_2}^{(N)} \otimes \mathrm{d}z_r^{(N)}\right) \in C_T^1(\mathbb{R}^K) \times C_{2,T}^1(\mathbb{R}^{K\times K})$$

$$\text{such that} \quad \lim_{N\to\infty} \sup_{0\le s<t\le T} \frac{|\delta z_{st} - \delta z_{st}^{(N)}|}{|t-s|^{\alpha}} + \sup_{0\le s<t\le T} \frac{|\mathbb{Z}_{st} - \mathbb{Z}_{st}^{(N)}|}{|t-s|^{2\alpha}} = 0.$$

We denote by $\mathscr{C}_{g,T}^{\alpha}(\mathbb{R}^K)$ the space of geometric rough paths.

Examples include Stratonovich Brownian motion, fractional Brownian motion, and Gaussian processes with sufficient time-correlation decay on a set of full probabilty measure.

Controlled rough paths and the rough integral

Definition

For a given Fréchet space E and $\mathbf{Z} \in \mathscr{C}^{\alpha}_{g,T}(\mathbb{R}^K)$, we let $\mathscr{D}_{Z,T}(E)$ denote the space of $\mathbf{Y} = (Y, Y') \in C^{\alpha}_{T}(E) \times C^{\alpha}_{T}(E^K)$ such

$$p(\delta Y_{st} - Y'_s \delta Z_{st}) = O(|t-s|^{2\alpha}),$$

for every seminorm p of E, endowed with the obvious topology.

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Theorem (Young/Sewing)

There exists a (unique) continuous linear map $\mathbf{I}_{\mathbf{Z}} : \mathscr{D}_{Z,T}(E^K) \to \mathscr{D}_{Z,T}(E)$ *such that* $\mathbf{I}_{\mathbf{Z}}(\mathbf{Y}) = (\int_0^{\cdot} \mathbf{Y} d\mathbf{Z}, Y)$ *where* $\int_0^0 \mathbf{Y} d\mathbf{Z} = 0_E$ *and*

$$p\left(\int_s^t \mathbf{Y}_r \, \mathrm{d}\mathbf{Z}_r - Y_s \delta Z_{st} - Y'_s \mathbb{Z}_{st}\right) = O(|t-s|^{3\alpha}).$$

Truly rough path

Definition

Let us call a path $\mathbf{Z} \in \mathscr{C}^{\alpha}_{g,T}(\mathbb{R}^K)$ truly rough if for all $s \in [0, T]$ and t in a dense set

$$\limsup_{t \downarrow s} \frac{|\delta Z_{st}|}{|t-s|^{2\alpha}} = \infty.$$

Lemma

A decomposition of a path $X \in C^{\alpha}_{T}(E)$ of the form

$$X_t = X_0 + \int_0^t \beta_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\mathbf{Z}_s$$

is unique.

*

Manifolds and the tensor bundles

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- Let $\mathfrak{X}$ denote the space of smooth vector fields
- Let Ω^k denote the space of smooth alternating k-forms
- Let $\mathcal{T}_s^r$ denote the space of smooth r-contravariant and s-covariant tensors.
- For a given $\phi \in \text{Diff}$ and $\tau \in \mathcal{T}_s^r$, let $\phi^* \tau$ denote the pullback and $\phi_* \tau = (\phi^{-1})^* \tau$ denote the pushforward.

Lie derivative

The Lie derivative of $\tau \in \mathcal{T}^r_s$ along $u \in C_T(\mathfrak{X})$ is defined by

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=t} \phi^*_{rt}\tau = \pounds_{u_t}\tau, \quad \text{where} \quad \dot{\phi}_{ts} = u_t(\phi_{ts}), \quad \phi_{ss} = \mathrm{id}\,.$$

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For $f \in \Omega^0$, $u, v \in \mathfrak{X}$, $\alpha \in \Omega^1$, and non-vanishing $\mu \in \Omega^d$

$$\pounds_u f = \mathbf{d}f(u) = u^i_t \partial_{x^i} f, \quad \pounds_u v = [u, v] = (u^j \partial_{x^j} v^i - v^j \partial_{x^j} u^i)\partial_{x^i},$$

$$\pounds_u \alpha = (u^j \partial_{x^j}\alpha_i + \alpha_j \partial_{x^i} u^j)dx^i, \quad \pounds_u \mu = (\mathrm{div}_\mu\, u)\mu.$$

Rough flow on M

Theorem

There exists a continuous map

$$\mathrm{Flow} : C_T^\alpha(\mathfrak{X}) \times C_T^\infty(\mathfrak{X}^K) \times \mathscr{C}_{g,T}(\mathbb{R}^K) \to C_{2,T}^\alpha(\mathrm{Diff})$$

such that $\eta = \mathrm{Flow}(u, \xi, \mathbf{Z})$ *satisfies*

- $$\eta_{ts} = \eta_{t\theta} \circ \eta_{\theta s}, \quad \forall (t,s) \in [0,T]^2;$$
- *for all* $(s,m) \in [0,T] \times M$, $\eta_{\cdot s}(m) \in C^\alpha([s,T]; M)$ *is the unique solution of*

$$\mathrm{d}\eta_{ts}(m) = u_t(\eta_{ts}(m))\,\mathrm{d}t + \xi_t(\eta_{ts}(m))d\mathbf{Z}_t, \quad \phi_{ss}(m) = m \in M;$$

Theorem

There exists a continuous map

$$\mathrm{Flow} : C_T^\alpha(\mathfrak{X}) \times C_T^\infty(\mathfrak{X}^K) \times \mathscr{C}_{g,T}(\mathbb{R}^K) \to C_{2,T}^\alpha(\mathrm{Diff})$$

such that $\eta = \mathrm{Flow}(u, \xi, \mathbf{Z})$ *satisfies*

- $$\eta_{ts} = \eta_{t\theta} \circ \eta_{\theta s}, \quad \forall (t,s) \in [0,T]^2;$$
- *for all* $(s,m) \in [0,T] \times M$, $\eta_{\cdot s}(m) \in C^\alpha([s,T]; M)$ *is the unique solution of*

$$\mathrm{d}\eta_{ts}(m) = u_t(\eta_{ts}(m))\,\mathrm{d}t + \xi_t(\eta_{ts}(m))d\mathbf{Z}_t, \quad \phi_{ss}(m) = m \in M;$$

that is, for all smooth $f : M \to \mathbb{R}$,

$$\Leftrightarrow \quad \eta_{ts}^* f(m) = f(m) + \int_s^t \eta_{rs}^* \pounds_{u_r} f(m) dr + \int_s^t \eta_{rs}^* \pounds_{\xi_r} f(m) d\mathbf{Z}_r.$$

The Lie chain rule

Lemma

- *Let* $\beta \in C_T(\mathcal{T}^r_s)$, $\boldsymbol{\sigma} = (\sigma, \sigma') \in \mathfrak{D}_{\mathbf{Z},T}((\mathcal{T}^r_s)^K)$, *and*

$$\tau_t = \tau_0 + \int_0^t \beta_s \,\mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}\mathbf{Z}_s, \quad t \in [0,T].$$

Then

$$\eta^*_{ts}\tau_t = \tau_s + \int_s^t \eta^*_{rs}\left(\beta_r + \pounds_{u_r}\tau_r\right)\,\mathrm{d}r + \int_0^t \eta^*_{rs}\left(\sigma_r + \pounds_{\xi_r}\tau_r\right) d\mathbf{Z}_r .$$

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- *For a given* $\tau_0 \in \mathcal{T}^r_s$, $\tau_t = \eta_{t0*}\tau_0$ *satisfies*

$$\tau_t + \int_0^t \pounds_{u_s}\tau_s \, \mathrm{d}s + \int_0^t \pounds_{\xi_s}\tau_s \, \mathrm{d}\mathbf{Z}_s = \tau_0.$$

Momentum and the coadjoint operator

- Recall that if $G = \mathrm{Diff}$, then $\mathfrak{g} = \mathfrak{X}$. We want a canonical characterization of $\mathfrak{g}^*$, where 'momentum' $\delta\ell/\delta\mu$ will lie.

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- Let $\mathfrak{X}^\vee = \Omega^1 \otimes \Omega^d$ and $\langle\cdot,\cdot\rangle_{\mathfrak{X}} : \mathfrak{X}^\vee \times \mathfrak{X} \to \mathbb{R}$ be defined by

$$\langle \alpha \otimes \mu, u \rangle_{\mathfrak{X}} = \int_M \alpha(u)\mu, \quad \alpha \otimes \mu \in \mathfrak{X}^\vee, \;\; u \in \mathfrak{X};$$

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- Noting that $\mathrm{ad}_u\, v = -[u, v] = -\pounds_u v$, for all $\alpha \otimes \mu \in \mathfrak{X}^\vee$ and $u \in \mathfrak{X}$, we find

$$\langle \alpha \otimes \mu, \mathrm{ad}_u\, v \rangle_{\mathfrak{X}} = \langle \pounds_u(\alpha \otimes \mu), v \rangle,$$

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$$\mathrm{ad}_u^* = \pounds_u.$$

- In the incompressible setting on a Riemannian manifold (M, g), we define the dual of the μ_g-divergence-free vector fields $\mathfrak{X}_{\mu_g}$ to be

$$\mathfrak{X}_{\mu_g}^\vee = \mathfrak{X}^\vee / \mathbf{d}\Omega^0 \otimes \mu_g.$$

Advected variables and the diamond operator

- Let $\mathfrak{A}$ be a direct summand of tensors field bundles and $\mathfrak{A}^\vee$ denote the canonical dual. Denote the duality pairing by $\langle \cdot, \cdot \rangle_{\mathfrak{A}} : \mathfrak{A}^\vee \times \mathfrak{A} \to \mathbb{R}$.

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- Paths in $\mathfrak{A}$ will model variables like temperature, density, or buoyancy.
- Define $\diamond : \mathfrak{A}^\vee \times \mathfrak{A} \to \mathfrak{X}^\vee$ by

$$\langle b, \pounds_u a \rangle_{\mathfrak{A}} = -\langle b \diamond a, u \rangle_{\mathfrak{X}} \quad \forall a \in \mathfrak{A},\ b \in \mathfrak{A}^\vee,\ u \in \mathfrak{X}$$

The Lagrangian

- Let $\ell : \mathfrak{X} \times \mathfrak{A} \to \mathbb{R}$. Assume that $\frac{\delta \ell}{\delta u} : \mathfrak{X} \times \mathfrak{A} \to \mathfrak{X}^\vee$ and $\frac{\delta \ell}{\delta a} : \mathfrak{X} \times \mathfrak{A} \to \mathfrak{A}^\vee$ are continuous:

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \ell(u + \epsilon \delta u, a + \epsilon \delta a) = \langle \frac{\delta \ell}{\delta u}(u, a), \delta u \rangle_{\mathfrak{X}} + \langle \frac{\delta \ell}{\delta a}(u, a), \delta a \rangle_{\mathfrak{A}}$$

- Assume that $\frac{\delta \ell}{\delta u}(\cdot, a) : \mathfrak{X} \to \mathfrak{X}^\vee$ is an isomorphism for all $a \in \mathfrak{A}$.

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- Assume that $\frac{\delta \ell}{\delta u}(\cdot, a) : \mathfrak{X} \to \mathfrak{X}^\vee$ is an isomorphism for all $a \in \mathfrak{A}$.
- Density $D = \rho \mu_g \in \mathfrak{A} = \Omega^d$ is an advected variable. Let

$$\ell(u, D) = \frac{1}{2} \int_M \rho g(u, u) \mu_g = \frac{1}{2} \int_M \rho u^\flat(u) \mu_g = \frac{1}{2} \langle u^\flat \otimes \rho \mu_g, u \rangle_{\mathfrak{X}}.$$

Then

$$\frac{\delta \ell}{\delta u} = u^\flat \otimes \rho \mu_g \in \mathfrak{X}^\vee, \quad \frac{\delta \ell}{\delta D} = \frac{1}{2} g(u, u) \in \mathfrak{A}^\vee = \Omega^0.$$

Hamilton-Pontryagin (HP) variational principle: flows

- Let $\mathbf{Z} \in \mathscr{C}_{g,T}(\mathbb{R}^K)$ be truly rough and

$$\mathrm{Diff}_{\mathbf{Z}} = \mathrm{Flow}(C^\alpha_T(\mathfrak{X}), C^\infty_T(\mathfrak{X}^K), \mathbf{Z})_{\cdot 0}.$$

- A given $\eta = \mathrm{Flow}(v, \sigma, \mathbf{Z}) \in \mathrm{Diff}_{\mathbf{Z}}$ satisfies

$$d\eta_t = v_t(\eta_t)\,\mathrm{d}t + \sigma_t(\eta_t)\mathrm{d}\mathbf{Z}_t, \quad \eta_0 = \mathrm{id}\,.$$

- For given $\lambda \in \mathscr{D}_{Z,T}(\mathfrak{X}^\vee)$, define

$$\int_0^T \langle \lambda_t, \mathrm{d}\eta_t \eta_t^{-1} \rangle_{\mathfrak{x}} := \int_0^T \langle \lambda_t, v_t \rangle_{\mathfrak{x}}\,\mathrm{d}t + \int_0^T \langle \lambda_t, \sigma_t \rangle_{\mathfrak{x}}\,\mathrm{d}\mathbf{Z}_t.$$

HP variational principle: functional and constraints

- Define the action functional $S^{HP_{\mathbf{Z}}} : \mathrm{Diff}_{\mathbf{Z}} \times C^{\alpha}_{T}(\mathfrak{X}) \times \mathfrak{D}_{Z,T}(\mathfrak{X}^{\vee}) \to \mathbb{R}$ by

$$S^{HP_{\mathbf{Z}}}(\eta, u, \lambda) = \int_0^T \ell(u_t, \eta_{t*}a_0)\mathrm{dt} + \langle \lambda_t, d\eta_t\eta_t^{-1} - u_t\mathrm{dt} - \xi \mathrm{d}\mathbf{Z}_t \rangle_{\mathfrak{X}}.$$

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- The Lagrange constraint imposes

$$\mathrm{d}\eta_t = u_t(\eta_t)\mathrm{d}t + \xi(\eta_t)\mathrm{d}\mathbf{Z}_t$$

and hence by the Lie chain rule

$$\mathrm{d}a_t + \pounds_{u_t} a_t \mathrm{d}t + \pounds_{\xi} a_t \mathrm{d}\mathbf{Z}_t = 0.$$

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- The Clebsch variational principle, which I am not presenting, directly imposes the advection relation. We can avoid the assumption of truly roughness in this case due to the fundamental theorem of rough calculus of variations.

HP variational principle: variations

- For a given $\delta w \in C_T^\infty(\mathfrak{X})$ with $\delta w_0 = \delta w_T = 0$, define $\psi : (-1,1) \in [0,T] \longrightarrow \text{Diff}$ by

$$\dot{\psi}_t^\epsilon(m) = \epsilon \dot{\delta w}_t(\psi_t^\epsilon(m)), \quad \psi_0^\epsilon(m) = m.$$

and $\eta_t^\epsilon = \psi_t^\epsilon \circ \eta_t$ and it can be shown that

$$d\eta_t^\epsilon = \left(\dot{\delta w}_t + \text{ad}_{v_t}\, \delta w_t\right) \text{dt} + \text{ad}_{\sigma_t}\, \delta w_t \text{d}\mathbf{Z}_t.$$

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- We take variations of u and λ of the form $u_\epsilon = u + \epsilon \delta u$ and $\lambda_\epsilon = \lambda + \epsilon \delta \lambda$ with

$$\delta u_0 = \delta u_T = 0 \quad \text{and} \quad \delta \lambda_0 = \delta \lambda_T = 0.$$

HP variational principle

Theorem ([Crisan et al., 2020b])

A curve (η, u, λ) *is a critical point of* $S^{HP\mathbf{z}}$ *iff for all* $[0, T]$*,*

$$
\begin{aligned}
&\mathrm{d}\eta_t = u_t(\eta_t)\mathrm{d}t + \xi(\eta_t)\mathrm{d}\mathbf{Z}_t, \quad t \in (0, T], \quad \eta_0 = \mathrm{id}, \\
&m = \frac{\delta \ell}{\delta u}(u, a) = \lambda, \\
&m_t + \int_0^t \pounds_{u_s} m_s \mathrm{d}s + \int_0^t \pounds_\xi m_s \mathrm{d}\mathbf{Z}_s \overset{\mathfrak{X}^\vee}{=} m_0 + \int_0^t \frac{\delta \ell}{\delta a}(u_s, a_s) \diamond a_s \mathrm{d}s, \\
&a_t + \int_0^t \pounds_{u_s} a_s \mathrm{d}s + \int_0^t \pounds_\xi a_s \mathrm{d}\mathbf{Z}_s \overset{\mathfrak{A}}{=} a_0, \quad a_t = \eta_{t*} a_0.
\end{aligned}
$$

A Kelvin circulation theorem

Recall that density $D \in C_T^\alpha(\Omega^d)$ is an advected variable:

$$\mathrm{d}D + \pounds_u D\mathrm{d}t + \pounds_\xi D\mathrm{d}\mathbf{Z}_t = 0 \quad \Leftrightarrow \quad D_t = \eta_{t*}D_0.$$

Moreover, $\frac{\delta \ell}{\delta u}, \frac{\delta \ell}{\delta a} \in \Omega^1 \otimes \Omega^d$, so that $\frac{1}{D}\frac{\delta \ell}{\delta u}, \frac{1}{D}\frac{\delta \ell}{\delta a} \in \Omega^1$ are one-forms.

Theorem

Let Γ denote a compactly embedded one-dimensional smooth submanifold of M. If D_0 is non-vanishing, then

$$\int_{\eta_t\Gamma} \frac{1}{D_t}\frac{\delta \ell}{\delta u}(u_t, a_t) = \int_\Gamma \frac{1}{D_0}\frac{\delta \ell}{\delta u}(u_0, a_0) + \int_0^t \int_{\eta_s\Gamma} \frac{1}{D_s}\frac{\delta \ell}{\delta a}(u_s, a_s) \diamond a_s \mathrm{d}s.$$

Incompressible Euler

Define the Lagrangian $\ell : \mathfrak{X}_{\mu_g} \times \mathfrak{A} \to \mathbb{R}$ by

$$\ell(u, D = \rho\mu_g) = \frac{1}{2}\int_M \rho g(u,u)\mu_g.$$

Applying $\frac{1}{\rho\mu_g}$ to the momentum equation (i.e., $\frac{\delta\ell}{\delta u}$) we get

$$\mathrm{d}u_t^\flat + \pounds_{u_t} u_t^\flat \mathrm{d}t + \pounds_\xi u_t^\flat \mathrm{d}\mathbf{Z}_t \overset{\Omega^1}{=} \frac{1}{2}\mathbf{d}g(u_t, u_t)\mathrm{d}t - \frac{1}{\rho_t}\mathbf{d}p_t \mathrm{d}t - \frac{1}{\rho_t}\mathbf{d}q_t \mathrm{d}\mathbf{Z}_t,$$

$$\mathbf{d}^* u^\flat = 0 = \operatorname{div} u,$$

$$\mathrm{d}\rho_t + \pounds_{u_t}\rho_t \mathrm{d}t + \pounds_\xi \rho_t \mathrm{d}\mathbf{Z}_t \overset{\Omega^0}{=} 0.$$

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$$\begin{aligned} \mathrm{d}u_t^\flat + \pounds_{u_t} u_t^\flat \mathrm{d}t + \pounds_\xi u_t^\flat \mathrm{d}\mathbf{Z}_t &\overset{\Omega^1}{=} \frac{1}{2}\mathbf{d}g(u_t,u_t)\mathrm{d}t - \frac{1}{\rho_t}\mathbf{d}p_t\mathrm{d}t - \frac{1}{\rho_t}\mathbf{d}q_t\mathrm{d}\mathbf{Z}_t, \\ \mathbf{d}^* u^\flat &= 0 = \operatorname{div} u, \\ \mathrm{d}\rho_t + \pounds_{u_t}\rho_t \mathrm{d}t + \pounds_\xi \rho_t \mathrm{d}\mathbf{Z}_t &\overset{\Omega^0}{=} 0. \end{aligned}$$

In the special case $\rho \equiv 1$ (homogeneous fluid), we find

$$\mathrm{d}u_t + \pounds_{u_t} u_t^\flat \mathrm{d}t + \pounds_\xi u_t^\flat \mathrm{d}\mathbf{Z}_t \overset{\Omega^1}{=} \frac{1}{2}\mathbf{d}g(u_t,u_t)\mathrm{d}t - \mathbf{d}p_t\mathrm{d}t - \mathbf{d}q_t\mathrm{d}\mathbf{Z}_t.$$

Solution properties of Euler's equations

Theorem ([Crisan et al., 2020a])

Let $m > d/2 + 1$. For given $\{\xi_k\}_{k=1}^K \in C_{\mathrm{div}}^{m+3}$ and $u_0 \in W_{\mathrm{div}}^{m,2}$, there exists a unique local maximal time $T^* = T(u_0, \xi, \mathbf{Z})$ and solution $(u, p) \in C_{T^*} W_{\mathrm{div}}^{m,2} \times C_{T^*}^{\alpha} W^{m-3,2}$ of the homogeneous rough Euler system. Moreover, if $T^* < \infty$,

$$\int_0^{T^*} |\omega_t|_{L^\infty} dt = +\infty,$$

where $\omega = \mathbf{d}u^\flat$ ($\omega = \mathrm{curl}\, u$).

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Corollary

If $d = 2$, then for all $p \geq 2$, vorticity is conserved $|\omega_t|_{L^p} = |\omega_0|_{L^p}$ and hence $T^* = \infty$.

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- Add data directly to varitiational principle

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